Steel column base classification

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The influence of the rotational characteristics of the column bases on the structural frame response is discussed and specific design criteria for stiffness classification into semi-rigid and rigid joints are derived. The particular case of an industrial portal frame is then considered.

Key words: Column bases, stiffness classification, sway frames, non-sway frames, Eurocode 3

1 Introduction

The actual mechanical properties of structural joints are known to significantly influence the behaviour of building frames. The correct evaluation of the mechanical properties of the joints, in terms of stiffness, strength and ductility, may therefore be considered as a key aspect in any structural frame analysis and design process. This is recognised by modern codes as Eurocode 3 [1]. In this code, design rules for several joint configurations are provided, as well as so-called classification boundaries which allow to define whether the beam-to-column joints are:

- Stiffness classification

Pinned, semi-rigid or rigid

- Strength classification

Pinned, partial-strength or full-strength

Some information on ductility aspects is also provided.
Few rules for the characterisation and the classification [2 to 4] of column bases being available, an ad-hoc working group was set-up at European level. In the present paper, the investigations [5, 6] carried out by the ad-hoc working group in the field of classification are presented and stiffness classification criteria are proposed. As for beam-to-column joints [7], a distinction is made between column bases belonging respectively to sway and non-sway frames.

2 Column bases in non-sway frames

A modification of the actual moment-rotation characteristic of column bases is likely to affect the whole response of non-sway frames, and in particular the lateral displacements of the beams and the buckling resistance of the column. This second aspect - the buckling resistance of the columns - is the one for which the influence is rather important, as seen in Figure 1, see [7]. It shows how the buckling length coefficient of a column pinned at the upper extremity is affected by the variation of the column base rotational stiffness. The buckling length coefficient K is reported on the vertical axis and is expressed as the square root of the ratio between the elastic critical load ($F_{cr.pin}$) of the column pinned at both extremities and that of the same column but restrained by the column base at the lower extremity ($F_{cr.res}$); it is seen to vary from 1,0 (pinned - pinned support conditions) to 0,7 (pinned - fixed support conditions).

$$K = \sqrt{\frac{F_{cr.pin}}{F_{cr.res}}} \tag{1}$$

$$F_{cr.pin} = \frac{\pi^2 E I_c}{L_c^2} \tag{2.a}$$

$$F_{cr.res} = \frac{\pi^2 E I_c}{(K L_c)^2} \tag{2.b}$$

where E is the modulus of elasticity of steel; L_c and I_c are respectively the system length and the moment of inertia of the column. In Figure 1, the non-dimensional stiffness $\overline{S}$ of the column base is reported in a logarithmic scale on the horizontal axis.

$$\bar{S} = \frac{S_{j.ini}\, L_c}{EI_c} \qquad (3)$$

$S_{j.ini}$ is the initial elastic stiffness in rotation of the column base. The numerical values indicated in Figure 1 have been obtained by considering the particular case of a 4 m length column with an HE 200 B cross-section.

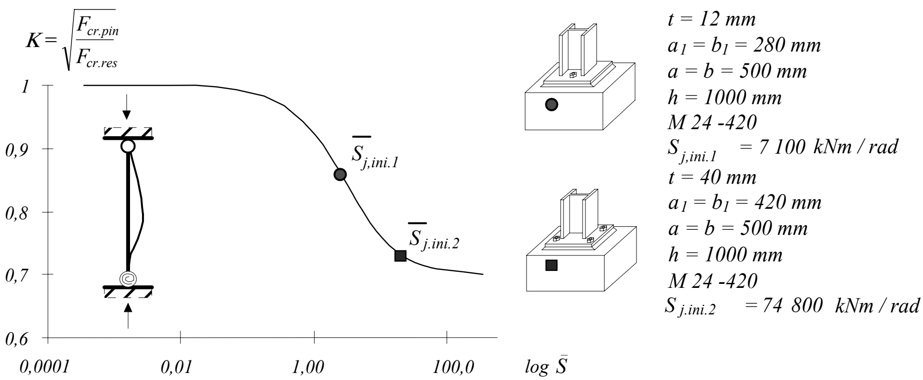


Figure 1: Elastic critical buckling load versus the initial stiffness of the column base [7]

The actual initial stiffness of two typical column bases is reported in Figure 1:

- column base with a base plate and two anchor bolts inside the H cross-section; this configuration is traditionally considered as pinned, but possesses an initial stiffness $S_{j.ini.1}$ equal to 7100 kNm/rad;
- column base with a base plate and four bolts outside the column cross-section; such a column base is usually considered as rigid even if its actual initial stiffness $S_{j.ini.2}$ = 74800 kNm/rad is not infinite.

As for beam-to-column joints, it may be concluded that stiff column bases always deform slightly in rotation while presumably pinned ones exhibit a non-zero rotational stiffness. Some column bases are however so flexible or so rigid, that the structural frame response obtained by considering the actual column base characteristics in rotation is not significantly different from that obtained by modelling respectively the column bases as perfectly pinned or rigid. For beam-to-column joints, this has led to the concept of stiffness classification into pinned, semi-rigid and rigid joints, see [1].

The stiffness classification in Eurocode 3 Annex J is achieved by comparing the initial stiffness of the beam-to-column joints to boundary values. For instance, rigid joints are characterised by a stiffness higher than $8\,EI_b/L_b$ where I_b and L_b are respectively the moment of inertia and the length of the beam. This check is based on a so-called "5% criterion". It says that a joint may be considered as rigid if the ultimate resistance of the frame in which it is incorporated is not affected by more than 5% in comparison with the situation where fully rigid joints are considered, see [8].

By adopting the same basic principle a rigid classification boundary for column bases may be derived, but with different levels of sophistication as shown in sections 2.1 and 2.2.

2.1 *Simple derivation of classification boundaries*

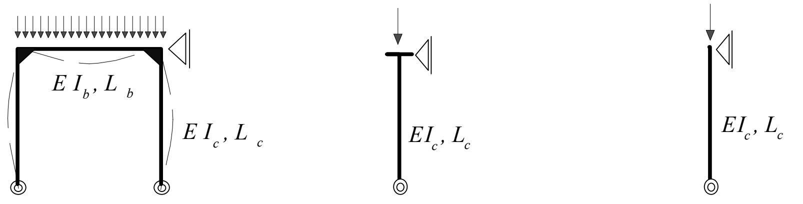
The single storey - single bay non-sway frame shown in Figure 2.a is considered. The study of the sensitivity of this frame to a variation of the column base stiffness properties is influenced by the beam and column characteristics in bending, EI_b/L_b and EI_c/L_c respectively. Two limit cases are however obtained when:

- the beam is rather stiff and a rigid joint connects the column to this beam, as shown in Figure 2.b ($EI_b/L_b = \infty$);
- the beam is rather flexible (or when a pinned joint connects the beam to the column); this situation is illustrated in Figure 2.c ($EI_b/L_b = 0$).

The application of the "5% criterion" to the first limit case (column fixed at top extremity), with the objective to derive a classification boundary for rigid column bases, writes as follows:

$$\frac{\dfrac{\pi^2 EI_c}{(KL_c)^2}}{\dfrac{\pi^2 EI_c}{(0{,}5\,L_c)^2}} \geq 0{,}95 \qquad (4)$$

For sake of simplicity, it is applied to the critical elastic loads and not to the ultimate ones (integrating the effects of plasticity, imperfections, ...); by doing so, a safe value of the rigid stiffness boundary is obtained as the effect of a modification of the column end restraints always results in a lower variation of the ultimate carrying capacity than of the elastic critical one.



a) Portal frame (non-sway) b) Isolated column (top-fixed) c) Isolated column (top-pinned)

Figure 2: Non-sway portal frame and isolated columns for classification study

From Equation (4), the minimum value of the buckling length coefficient may be derived:

$$K \leq 0,513 \tag{5}$$

According to the Annex E on the effective buckling length of members in compression [1], the K coefficient is expressed as a function of flexibility coefficients (k_l , k_u) at both column ends and writes for non-sway frames:

$$K = \frac{1+0,145\,(k_l+k_u)-0,265\,k_l\,k_u}{2-0,364\,(k_l+k_u)-0,247\,k_l\,k_u} \tag{6}$$

with:

$$k_l = \frac{EI_c\,/\,L_c}{EI_c\,/\,L_c + S_{j,ini}} \text{ at lower extremity} \tag{7.a}$$

$$k_u = 0 \text{ at upper extremity} \tag{7.b}$$

From Equations (5) to (7), the minimum value of the elastic initial stiffness $S_{j.ini}$ that presumably rigid column bases have to exhibit may be derived as follows:

$$S_{j.ini} \geq 48\;EI_c\,/\,L_c \tag{8}$$

A similar approach may be followed for the second limit case (Fig. 2.c) and the following boundary is extracted

$$S_{j.ini} \geq 40\;EI_c\,/\,L_c \tag{9}$$

This limit is less restrictive than the first one; this shows that:

- the stiffness requirement is system dependent;
- the stronger requirement on joint stiffness is obtained for the first limit case where the flexibility at the extremities of the column is strictly resulting from that of the column base.

This is physically understandable and, as a consequence, the following simple stiffness classification boundary to distinguish between rigid and semi-rigid column bases is suggested:

$$\text{Rigid column bases } S_{j.ini} \geq 48\, EI_c / L_c \quad (10.a)$$

$$\text{Semi-rigid column bases } S_{j.ini} < 48\, EI_c / L_c \quad (10.b)$$

2.2 *Precise derivation of classification boundaries*

The "5% criterion" may be more accurately applied to the first limit case (column fixed at top extremity) by referring to the ultimate frame resistance and not to the elastic one as shown in section 2.1. As a valuable approximation, the ultimate column resistance N_u may be expressed as [9, 10]:

$$\frac{1}{N_u} = \frac{1}{N_p} + \frac{1}{N_{cr}} \quad (11)$$

where N_p and N_{cr} designate respectively the squash load and the critical elastic buckling load of the column. As $N_{cr} = N_p / \bar{\lambda}^2$, $\bar{\lambda}$ being the reduced slenderness of the column, Equation (11) writes also:

$$N_u = N_p \frac{1}{1+\bar{\lambda}^2}\,. \quad (12)$$

With reference to Figure 2.b, the reduced slenderness of the column with a fully stiff column base at lower end equals:

$$\bar{\lambda}_{stif} = 0,5\,\bar{\lambda}_{pin} \tag{13.a}$$

while that of the column with a semi-rigid column base writes:

$$\bar{\lambda}_{res} = K\,\bar{\lambda}_{pin} \tag{13.b}$$

where $\bar{\lambda}_{pin}$ is the reduced slenderness of the column assumed as pinned at both extremities (K = 1,0).

The application of the "5 % criterion" to the ultimate column resistance therefore gives N_p being considered as constant:

$$\frac{1+(0,5)^2\bar{\lambda}_{pin}^2}{1+K^2\bar{\lambda}_{pin}^2} \geq 0,95 \tag{14.a}$$

or

$$K \leq 0,513\sqrt{1+\frac{1}{5\bar{\lambda}_{pin}^2}} \tag{14.b}$$

Expression (14.b) may be compared to Equation (5). For high values of relative slenderness $\bar{\lambda}_{pin}$ both expressions are similar. In such cases, the ultimate resistance N_u equals N_{cr} and a high boundary value of $S_{j.ini}$ (see Formula 8) is required. For low values of $\bar{\lambda}_{pin}$, the condition (14.b) relaxes and, as a consequence, less severe boundary values of $S_{j.ini}$ are required, the influence of the cross-section yielding becoming then more predominant than the instability. In case of $\bar{\lambda}_{pin}$ = 0,48, Equation (14.b) writes $K \leq 0,7$, what means that any column base, even a perfectly pinned one, will be considered as rigid. The ultimate resistance N_u is then so close to the squash load N_p that no significant reduction of the resistance because of instability is contemplated. By integrating Formulae (14.b) into Formulae (6) to (7b), the following stiffness boundaries are obtained:

if $\bar{\lambda}_{pin} \leq 0{,}48$ $\qquad S_{j.ini} \geq 0$ (15.a) if $\overline{\lambda}_{pin} > 0{,}48$ $\quad S_{j,ini} \geq \frac{1{,}145 - 0{,}838\,\mu}{1{,}026\,\mu - 1} \frac{4\,EI_c}{L_c}$ (15.b)

with:

$$\mu = \sqrt{1 + \frac{1}{5\,\overline{\lambda}_{pin}^2}} \quad (16)$$

For practical applications, simpler expressions are proposed which fit rather well, as seen in Figure 3, with the exact ones in the usual range of application ($\overline{\lambda}_{pin} \leq 2$). These are

if $\overline{\lambda}_{pin} \leq 0{,}5$ $\quad S_{j.ini} \geq 0$ (17.a)

if $0{,}5 < \overline{\lambda}_{pin} < 3{,}93$ $\quad S_{j.ini} \geq 7(2\overline{\lambda}_{pin} - 1)EI_c / L_c$ (17.b)

if $\overline{\lambda}_{pin} \geq 3{,}93$ $\quad S_{j.ini} \geq 48\,EI_c / L_c$ (17.c)

As a safe approximation, Formulae (10.a) and (10.b) may obviously be applied for any column slenderness.

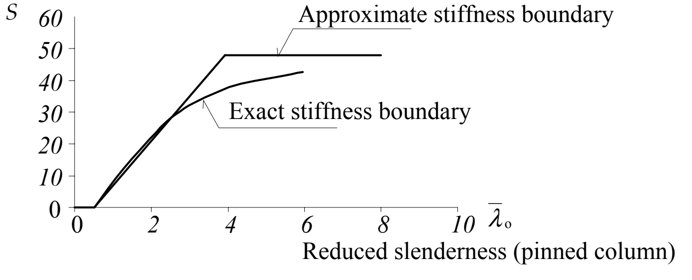


Figure 3: Exact and approximate stiffness boundaries for non-sway frames

The followed approach, which leads to Expressions (10) and (17), allows to classify the column bases according to the column properties only. A more precise boundary which would also depend on the k_u coefficient could obviously be derived but its application would be far more complicated.

A stiffness boundary allowing to distinguish simple joints from semi-rigid ones may be defined by referring to similar principles than those used in the previous pages. The value obtained is however so low that all actual column bases are practically classified as semi-rigid; therefore a semi-continuous modelling is always required. But practically, even if a joint is semi-rigid, nothing prevents the designer to consider it as pinned, as this is presently done in design, provided that the joint exhibits a sufficient rotational ductility. As a consequence, no pinned classification boundary is derived and proposed here.

3 Column bases in sway frames

The sway frames are more sensitive than non-sway ones to the variation of the rotational properties of column bases, mainly because of their high sensitivity to lateral deflections as well as to changes of the overall stability conditions when the lateral flexibility increases.

To illustrate this statement, a single-bay single-storey sway frame is considered in Figure 4. The diagram indicates the evolution of the ratio $\beta_s = \delta_{res} / \delta_{pin}$ (ratio between the lateral deflection δ_{res} of the frame with actual column base stiffness and the deflection δ_{pin} of the frame with assumed ideally pinned column bases) with increasing values of $\overline{S}$. The non-dimensional stiffness $\overline{S}$ defined by Eq. (3) is again reported in a logarithmic scale. First order elastic theory is used to compute the values of δ.

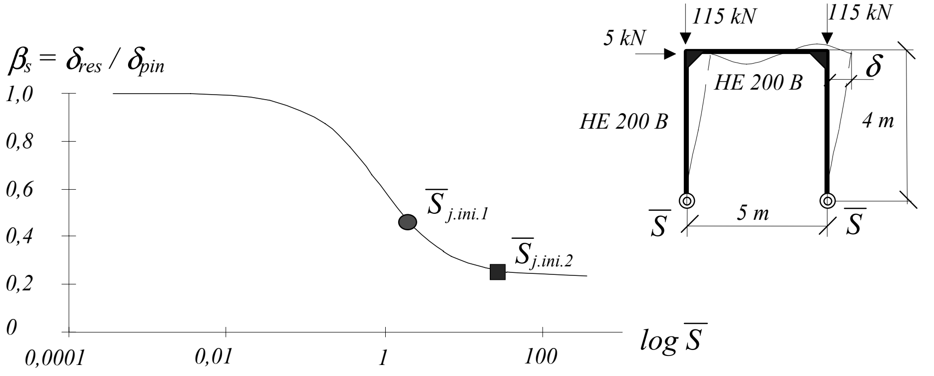


Figure 4: Sensitivity of the sway deflection to a variation of the column base stiffness in a portal frame

A stiffness classification boundary similar to that expressed in the case of non-sway frames may again be derived here on the basis of a "5% resistance criterion". For sway frames also it may be demonstrated that the more restrictive situation corresponds to the limit case where the beam flexural stiffness is rather high in comparison with that of the columns.

Referring to the isolated column represented in Fig. 5.b the classification boundary can be derived.

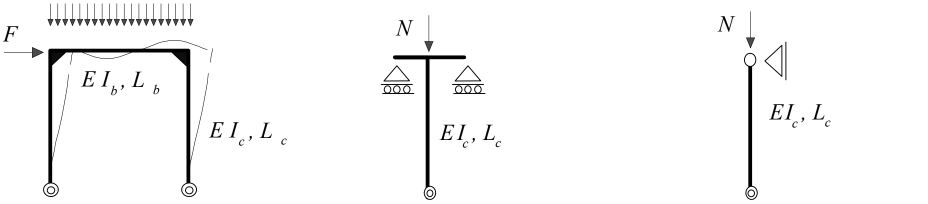


Figure 5: Sway portal frame and isolated columns for classification study

The application of the "5% resistance criterion" writes in this case:

$$\frac{\dfrac{\pi^2\, EI_c}{(KL_c)^2}}{\dfrac{\pi^2\, EI_c}{(L_c)^2}} \geq 0{,}95 \tag{18}$$

As a result:

$$K \leq 1{,}026 \tag{19}$$

For sway frames, the *K* - *k* relationship given by Equation (6) has to be replaced by the following one [1]:

$$K = \sqrt{\frac{1-0{,}2\,(k_l+k_u)-0{,}24\,k_l\,k_u}{1-0{,}8\,(k_l+k_u)+0{,}6\,k_l\,k_u}} \tag{20}$$

while Equations (7.a) and (7.b) remain unchanged. The combination of these equations leads to the following expression of the stiffness classification boundary:

$$S_{j.ini} \geq 11\, EI_c \,/\, L_c \tag{21}$$

However the 5% resistance criterion fully disregards the aspects of lateral frame deflections, which have been pointed out as important. The lateral deflection δ of the portal frame illustrated in Figure 5.a may by written, see Annex A [11]

$$\delta_{res} = \frac{F/2\ L_c^3}{E\ I_c}\frac{1}{12}\frac{4\ (3+\overline{S})+6\ (4+\overline{S})\ \varsigma}{\overline{S}+6\ (1+\overline{S})\ \varsigma} \tag{22}$$

where:

$$\varsigma = \frac{EI_b / L_b}{EI_c / L_c} \tag{23}$$

For $\overline{S} \to \infty$, the deflection for the frame with rigid column bases may be derived from (22):

$$\delta_{stif} = \frac{FL_c^3}{2\ EI_c}\frac{1}{12}\frac{4+6\varsigma}{1+6\varsigma} \tag{24}$$

In comparison with the case where rigid column bases are used, Equation (24), the actual frame - where the column bases possesses some degree of flexibility - will experience a larger deflection, Formula (22); this increase of the lateral displacement may be expressed in terms of percentage (100ω) as follows:

$$\frac{\delta_{res}}{\delta_{stif}} = 1+\omega \tag{25}$$

As far as classification is concerned, a "100ω %" displacement criterion may be suggested with the objective to limit the increase of the lateral displacement of the actual frame to 100ω % of the deflection evaluated in the case of rigid column bases. By combining the expressions (25) (22) and (24), the value of the minimum rotational stiffness that the column bases should exhibit to be considered as rigid from a displacement point of view is derived:

$$\overline{S} \geq \frac{12+24\ \varsigma - 6\varsigma\,(1+\omega)\,\dfrac{4+6\ \varsigma}{1+6\ \varsigma}}{(4+6\ \varsigma)\ \omega} \tag{26}$$

This condition is illustrated in Figure 6. The required stiffness is seen to be rather insensitive to the values of ς for significant values of ω. Conservatively the values obtained for ς = 0,0 may be selected, i.e.:

for 100ω = 20% , the following stiffness boundary is obtained $\overline{S} \geq 15$,

for 100ω = 10% $\overline{S} \geq 30$,

for 100ω = 5% $\overline{S} \geq 60$.

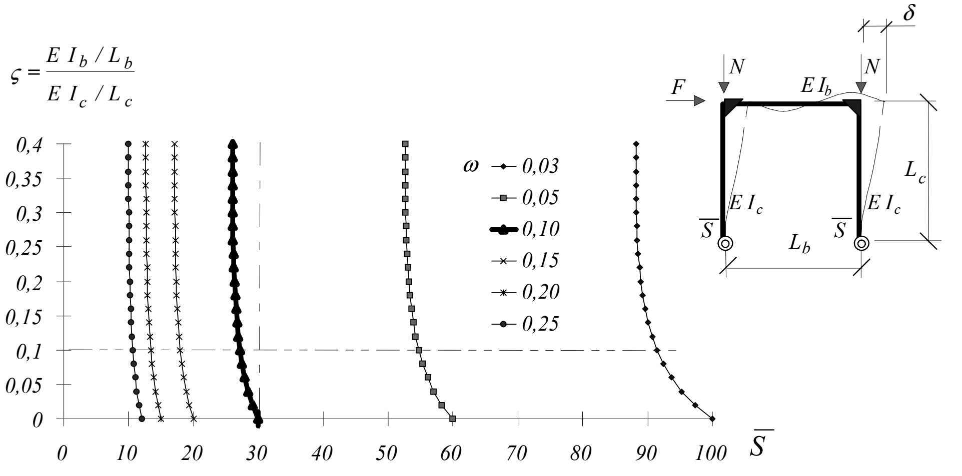


Figure 6: Displacement classification criteria for column bases in sway frames

As a consequence, the displacement classification criterion is seen to be much more restrictive than the resistance one given by Equation (21). The selection of the value for the boundary is obviously strongly related to the level of accuracy, which is thought to be necessary for the evaluation of the lateral frame deflection. A value of 10% appears to be quite realistic and the following stiffness classification boundary for presumably rigid column bases may be therefore proposed:

- rigid column bases in sway frames

$$S_{j.ini} \geq 30\, EI_c \,/\, L_c \tag{27.a}$$

- semi-rigid column bases in sway frames

$$S_{j.ini} < 30\, EI_c \,/\, L_c \tag{27.b}$$

For similar reasons than those given for non-sway frames, no classification boundary for presumably pinned column bases is suggested.

4 Summary and concluding remarks

Depending on the structural system (sway or non-sway frames), different rigid stiffness boundaries for the classification of column bases may be derived. The derivation of the boundaries is based on a sensitivity study of the structural system to the variation of the rotational stiffness properties of the column bases. The bases are assumed to be rigid as long as their actual behaviour in rotation does not influence the resistance of the frame at ultimate limit state by more than 5% and does not influence the lateral displacement under service loads by more than 10% [11].

The boundaries are shown in Figure 7 for non-sway and sway frames (for a particular value of $\bar{\lambda}_{pin}$ = 1,36) respectively. They are as follows:

- ***Non-sway frames***

  Rigid column bases:

  if $\bar{\lambda}_{pin} \leq 0{,}5$ $\quad S_{j.ini} \geq 0$ (28.a)

  if $0{,}5 < \bar{\lambda}_{pin} < 3{,}93$ $\quad S_{j.ini} \geq 7(2\bar{\lambda}_{pin} - 1)EI_c / L_c$ (28.b)

  if $\bar{\lambda}_{pin} \geq 3{,}93$ $\quad S_{j.ini} \geq 48\,EI_c / L_c$ (28.c)

  Semi-rigid column bases:

  if $\bar{\lambda}_{pin} \leq 0{,}5$ all joints rigid (29.a)

  if $0{,}5 < \bar{\lambda}_{pin} < 3{,}93$ $\quad S_{j.ini} < 7(2\bar{\lambda}_{pin} - 1)EI_c / L_c$ (29.b)

  if $\bar{\lambda}_{pin} \geq 3{,}93$ $\quad S_{j.ini} < 48\,EI_c / L_c$ (29.c)

- ***Sway frames***

  Rigid column bases:

  $S_{j.ini} \geq 30\,EI_c / L_c$ (30.a)

  Semi-rigid column bases:

  $S_{j.ini} < 30\,EI_c / L_c$ (30.b)

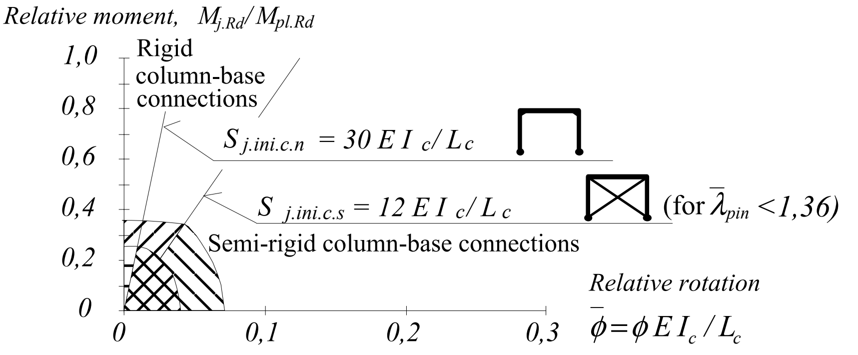


Figure 7: Proposed classification system according to the initial stiffness

The so-called ENV pre-normative version of Eurocode 3 [1] should be soon replaced by a full EN European Norm (Eurocode 3 EN1993), as a result of a long conversion period. In Eurocode 3 EN1993, all the design rules on structural joints and connections will be included into a single document (Part 1-8). The here-above listed stiffness classification boundaries for column bases are implemented in Part 1-8 and the present paper therefore provides background information to all potential users.

Acknowledgement

Within the framework of the European Project COST C1 (Semi-rigid behaviour of civil engineering structural connections) and the Technical Committee 10 of ECCS (European Convention for Constructional Steelwork) an ad-hoc working group prepared a background document on design of column bases for Eurocode 3. Members of this group are: D. Brown, SCI London; A.M. Gresnigt, TU Delft; J.P. Jaspart, University of Liège; Z. Sokol, CTU in Prague; J.W.B. Stark, TU Delft; C.M. Steenhuis, TU Eindhoven; J.C. Taylor, SCI London; F. Wald, CTU in Prague (convener of the group), K. Weynand, RTWH Aachen.

References

[1] Eurocode 3, ENV - 1993-1-1, Design of Steel Structures - General rules and roles for buildings, CEN, Brussels 1992, including Part 1.1, A2: Design of Steel Structures - General rules and roles for buildings, Annex J, European Prenorm, CEN, Brussels 1998.

[2] Bijlaard F. S. K., Nethercot D. A., Stark J. W. B., Tschemmernegg F., Zoetemeijer P.: Structural Properties of Semi-Rigid Joints in Steel Frames, IABSE Periodica, No.2, Survey report No. S-42/89, Basel 1989.

[3] Akiyama H.: Seismic Design of Steel Column for Architecture, in Japanese, Gibodoskupan, Tokyo 1996.

[4] King C. M., Owens G. W.: Plastic Design of Single Storey Pitched Roof Portals to EC3, in Steel Structures - Eurosteel 95. ed. Kounadis, Balkema, Rotterdam 1996, ISBN 90-5410-554-2, pp. 153 - 160.

[5] Wald F., Gresnigt A. M., Weynand K., J. P. Jaspart: Application of the Component Method to Column Bases, in COST C1 Conference Liege 1998, Brussels 1999, pp. 155 - 166, ISBN 92-828-6337-9.

[6] Column Bases in Steel Building Frames, COST C1, ed. K. Weynand, Brussels, 1999.

[7] Wald F., Seifert J.: The Column-Bases Stiffness Classification, in Nordic Steel Colloquium, Odense 1991, pp. 309 - 316.

[8] Bijlaard F., Steenhuis M: Prediction of the influence of connection behaviour on the strength, deformations and stability of frames, by classification of connections, Proceedings of the Second International Workshop on Connections in Steel Structures edit. by R. Bjorhovde, A. Colson, G. Haaijer and J. Stark, AISC, Chicago, 1992, pp. 307-318.

[9] Merchant W., The failure load of rigid jointed frameworks as influenced by stability, The Structural Engineer, Vol. 32, 1954, pp. 185-190.

[10] Massonnet C., Save M.: Calcul plastique des constructions, Volume 1: Structures dépendant d'un paramètre, Third Edition, Nelissen, Liège, Belgium, 1976.

[11] Wald F., Sokol Z.: Column Base Stiffness Classification, in Stability and Ductility of Steel Structures 1997, Nagoya 1997, pp. 675 – 682.

Annex A

In the case of lateral forces acting on the frame, see Figure A1, it is common to replace the structural system by a so-called equivalent frame. The restraining effect of the beam is transposed into a rotational spring κ acting at the top of the column. The characteristic κ of the "beam" spring is defined as follows

$$\kappa = M / \phi = M / (M \; L_b / 2 / (3 \, E \, I_b)) \tag{A1}$$

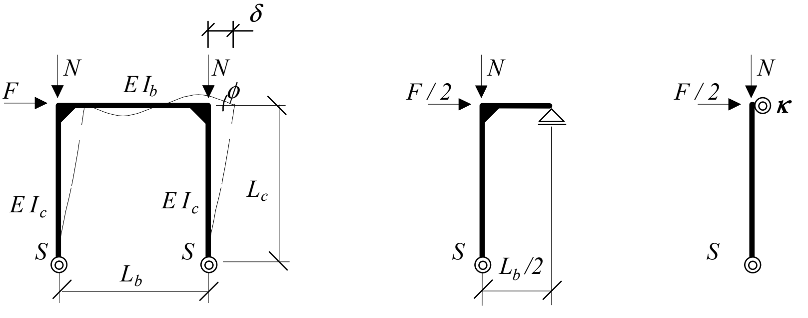


Figure A1: Equivalent structure for a portal frame

The study of the frame therefore reduces to that of a single column restrained at both extremities by two rotational springs, κ and S respectively. To express the behaviour of the equivalent frame, a matrix format may be adopted

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \phi \\ \delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ F/2 \end{Bmatrix} . \tag{A2}$$

For $\kappa = \infty$ and $S = \infty$, one has

$$K_{11} = \frac{4 EI_c}{L_c} + \frac{6 EI_b}{L_b} \, , \; K_{12} = K_{21} = \frac{6 EI_c}{L_c^2} \, , \; K_{22} = \frac{12 EI_c}{L_c^3} \, .$$

It is convenient to define

$$\bar{\phi} = \phi \, \frac{EI_c}{F / 2 L_c^2} \text{ and } \bar{\delta} = \delta \, \frac{EI_c}{F / 2 \, L_c^3} \, .$$

If $\kappa = \infty$ and the actual relative stiffness of the column base is considered, the K coefficients in Equation (A2) write

$$K_{11} = \frac{4EI_c}{L_c}\frac{3+\overline{S}}{4+\overline{S}} + \frac{6EI_b}{L_b}\ ,\ K_{12} = K_{21} = \frac{6EI_c}{L_c^2}\frac{2+\overline{S}}{4+\overline{S}}\ ,\ K_{22} = \frac{12EI_c}{L_c^3}\frac{1+\overline{S}}{4+\overline{S}}\ .$$

Equation (A2) may be rearranged as follows

$$\begin{bmatrix} 4\,\dfrac{3+\overline{S}}{4+\overline{S}} + 6\,\varsigma & 6\,\dfrac{2+\overline{S}}{4+\overline{S}} \\ 6\,\dfrac{2+\overline{S}}{4+\overline{S}} & 12\,\dfrac{1+\overline{S}}{4+\overline{S}} \end{bmatrix} \begin{Bmatrix} \overline{\phi} \\ \overline{\delta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{A3}$$

By solving this equation, the non-dimensional lateral deformation of the frame is derived:

$$\overline{\delta} = \frac{\delta EI_c}{F/2L_2^3} = \frac{1}{12}\frac{4(3+\overline{S})+6(4+\overline{S})\varsigma}{\delta EI_c + 6(1+\overline{S})\varsigma} \tag{A4}$$

From Equation (A4), the value of δ (= δ_{res}) may then be easily found.

Notations

k	coefficient
E	Young's modulus of steel
F	force
I	second moment of inertia
L	length
K	buckling length coefficient
N	normal force
$\overline{S}$	relative stiffness
δ	deformation
κ	stiffness of rotational spring
ζ	frame relative bending stiffness
$\overline{\lambda}$	reduced slenderness
ω	difference in percentage / 100

Subscripts

b	beam
c	column
cr	critical
j	joint
l	lower
ini	initial
n	non-sway
res	restrained
p	plastic
pin	pinned
s	sway
$stif$	stiffened
sem	semi-rigid
u	upper