Optimal number of load combinations in structural analysis

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In this paper, the optimal number of load combinations is calculated for analysing engineering structures. This number depends on the quantity of load cases, the cost of the structure and the cost of analysing one load combination. The problem can be reduced to a geometrical challenge: find polytopes that fit a sphere in hyperspace. Three examples show that the optimal number can be much larger than the number applied in current practice.

Keywords: Partial factors, LRFD, safety factors, load cases, load combinations, structural reliability, structural design, structural analysis, convex polytopes, circum-sphere, in-sphere

1 Introduction

For most structural types, the partial factors and load combinations are specified in codes of practice [1, 2]. They were determined based on good practice, theoretical research and engineering judgement. However, these specified load combinations are probably not optimal. Other combinations could guarantee an equally safe design while the structural cost might be less [3]. For example, the software CodeCal [4] can optimise partial factors

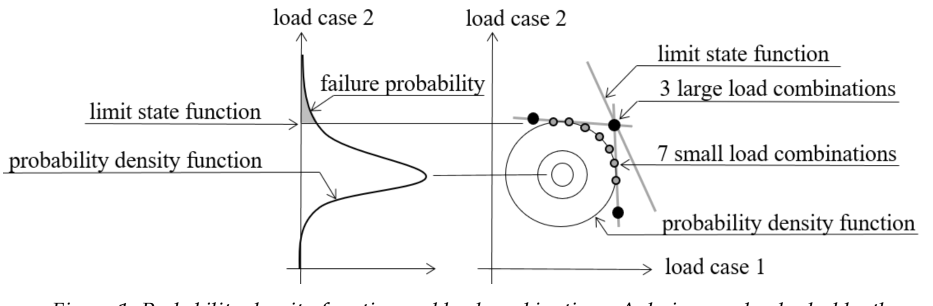

Figure 1. Probability density function and load combinations. A design can be checked by three large load combinations or by seven smaller load combinations. In both cases, accepted designs have sufficient safety.

for three load cases and two load combinations. When more load combinations are used the partial factors can be a bit smaller and the structural cost will be less (Fig. 1). On the other hand, analysing more load combinations gives extra design costs. In fact, there is a trade-off between structural costs and design costs. The optimal set of load combinations provides the minimal total cost.

2 Geometrical description

A load combination can be interpreted as a point in the space of load cases. Figure 2 gives an example of two load cases L and W and four load combinations. Only the first quadrant is plotted because partial factors always have positive values.

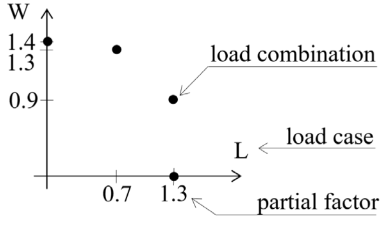

Figure 2. Load combinations in the space of load cases

A limit state function is the boundary in the space of load cases at which a structure fails. Failure is defined as not fulfilling a performance requirement, which can be related to the ultimate limit state or the serviceability limit state. For a well-designed structure, all load combinations are in the area enclosed by the limit state function. Many limit state functions are peace wise linear. Each linear part represents a failure mechanism. Moreover, for structures made of ductile components and joints, it can be proved that the limit state function is convex [5]. In theory, if a design would be structurally optimal the limit state function would be the convex envelope of the load combinations (Fig. 3).

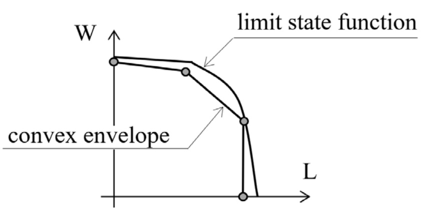

Figure 3. Limit state function and convex envelope of the load combinations

For two load cases a convex envelope of the load combinations is a polygon. For three load cases it is a polyhedron. For four or more load cases the convex envelope is a polytope [6].

In the space of load cases the probability density function can be plotted too (Fig. 4). The failure probability is the integral of the probability density outside the limit state function. For the failure probability to be small, the limit state function should be at sufficient distance from the origin. However, the structural cost is proportional to the values of the partial factors. For the structural cost to be small, the limit state function should be close to the origin.

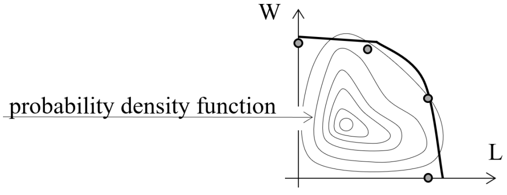

Figure 4. Contour plot of the probability density function of the load cases. Part of this function is outside the limit state function. The volume of this part is the failure probability.

The situation is idealised as follows (Fig. 5). Consider a sphere in a space of d dimensions. Middle point of the sphere is the origin and its radius is Ri. We consider part of the space for which all co-ordinate values are larger than or equal to zero. The sphere is enclosed by a polytope. This polytope is the convex hull of n points. A second sphere encloses the polytope. This sphere has a radius Re and its middle point is in the origin too. The challenge is to optimise the point co-ordinates such that Re/Ri obtains its smallest value (Fig. 6). The in-sphere represents a sufficiently safe limit state function while the circum-sphere represents the actual limit state function.

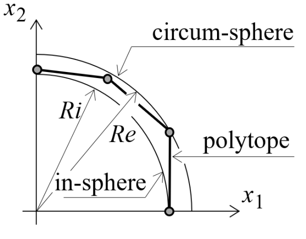

Figure 5. Geometrical model

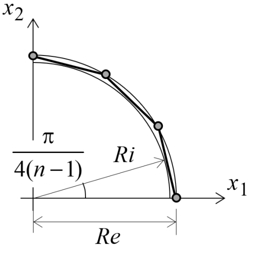

Figure 6. Optimal point positions for d = 2 and n = 4

$$\delta = \left.\frac{Re}{Ri}\right|_{\min} \tag{1}$$

For d = 2 this problem can be solved graphically (Fig. 6), the solution is

$$\delta = \frac{1}{\cos\frac{\pi}{4(n-1)}} \tag{2}$$

For $d \geq 3$, the problem is more complicated. It will be solved in Sections 4 and 5.

3 Optimal number of load combinations

The total construction cost C can be expressed as

$$C = \delta C_s + nC_a + C_o \,. \tag{3}$$

The term δC_s is the cost of the load carrying structure. It includes all costs that increase when the partial factors are increased in design, for example structural dimensions, material qualities, crane capacity and construction time. It can also include non-structural components. For example, when the floor thickness of a high-rise building is increased also the surface of the facade is increased. The factor C_s represents the structural costs if the design were evaluated by accurate probability analyses and were optimised in a number of design cycles (reliability based optimisation). The factor δ describes the extra safety margin – included in partial factors – that accounts for the finite number of load combination that is used instead of probability analysis. The definition of C_s is rather impractical, however, it can be estimated well as the structural cost of an already completed similar project.

The term nC_a is the cost of computer analysis of the structural design. It includes all costs that increase when the number of load combinations are increased in design, for example computer capacity, waiting time of the design team, checking the output and possible delays in construction. The cost of one analysis is C_a and it is repeated – by a computer – for all n load combinations.

The term C_o represents all other costs that do not depend on the partial factors or the number of load combinations.

We are looking for the minimal total cost C with respect to the number of load combinations n. Therefore, the following condition should hold.

$$\left.\frac{dC}{dn}\right|_{n=n_{\text{opt}}} = 0 \tag{4}$$

where n_{opt} is the optimal number of load combinations. Substitution of Eq. 3 in Eq. 4 gives

$$0 = \left.\frac{d\delta}{dn}\right|_{n=n_{opt}} C_s + C_a . \tag{5}$$

Substituting Eq. 2, which is for just two load cases, in Eq. 5 we obtain

$$0 = \frac{-\sin\dfrac{\pi}{4(n_{\text{opt}}-1)}}{-\cos^2\dfrac{\pi}{4(n_{\text{opt}}-1)}} \frac{-\pi}{4(n_{\text{opt}}-1)^2} C_s + C_a . \tag{6}$$

A closed form solution of n_{opt} does not exist. The solution can be accurately approximated by

$$\boxed{n_{\text{opt}} = 1 + \sqrt[3]{\frac{\pi^2}{16}\frac{C_s}{C_a}}} \tag{7}$$

which has an error less than 1. Eq. 7 is valid for two load cases $d = 2$.

4 Solutions for few load combinations

For small n, the solutions of δ as function of n and d are shown in Table 1. Each value δ represents a set of optimal point positions. The values have been analytically derived or were found by computer programs [7, 8]. As common in nonlinear optimisation it is not always certain whether the global minimum has been found. For $d = 3$ the result can be visually inspected (Fig. 7, Table 2). For $d \geq 4$ this is not possible and the true minimum could have been overlooked. Therefore, future improvements to the table can be expected. In Table 1, a simple rule can be observed for the accuracy of few load combinations.

An extra load case d can be compensated by two extra load combinations n.

For example, initially 6 load cases and 7 load combinations are used. If the number of load cases is increased to 7 the number of load combinations needs to be increased to 9 in order to obtain the same δ. If the partial factors do not change, the same structural safety is obtained.

Table 1. Margin δ as a function of n and d

		d, number of load cases														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	1	1.00														
	2	1.00	1.41													
	3	1.00	1.08	1.73												
	4	1.00	1.04	1.41	2.00											
	5	1.00	1.02	1.41	1.73	2.24										
	6	1.00	1.01	1.22	1.73	2.00	2.45									
n, number	7	1.00	1.01	1.13	1.73	1.78	2.24	2.65								
of load	8	1.00	1.01	1.13	1.41	1.78	2.00	2.45	2.83							
combinations	9	1.00	1.00	1.08	1.41	1.78	1.87	2.24	2.65	3.00						
	10	1.00	1.00	1.08	1.41	1.63		2.08	2.45	2.83	3.16					
	11	1.00	1.00	1.08	1.22	1.58			2.27	2.65	3.00	3.32				
	12	1.00	1.00	1.06	1.22	1.58				2.45	2.83	3.16	3.46			
	13	1.00	1.00	1.06	1.22	1.53				2.35	2.65	3.00	3.32	3.61		
	14	1.00	1.00	1.06	1.18	1.53	1.73				2.52	2.83	3.16	3.46	3.74	
	15	1.00	1.00	1.04	1.16	1.41	1.63				2.42	2.68	3.00	3.32	3.61	3.87

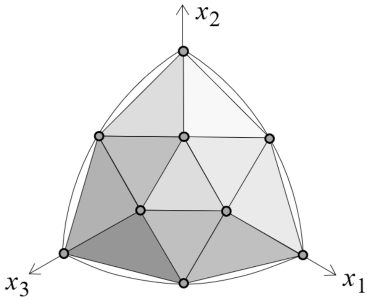

Figure 7. Optimal point co-ordinates for d = 3 and n = 9

Table 2. Optimal point co-ordinates for d = 3 and n = 9

	x_1	x_2	x_3
1	1.08239	0	0
2	0	1.08239	0
3	0	0	1.08239
4	0.76537	0.76537	0
5	0.76537	0	0.76537
6	0	0.76537	0.76537
7	0.91136	0.41291	0.41291
8	0.41291	0.91136	0.41291
9	0.41291	0.41291	0.91136

For large numbers of d and n, the computation of the optimal point co-ordinates requires much time. This is caused by the many relations that exist in multi-dimensional spaces. For example, for d = 10 and n = 100, 17310309456440 possible facets need to be processed to compute the in-radius. Moreover, this needs to be repeated for every of the thousands of changes in the point co-ordinates to find the optimal positions. It will take many years before computers can do this quickly.

The point coordinates in this paper are not the same as the sensitivity factors in probability theory. If they were, the calculation of partial factors would be easy [2, Annex E]. On the other hand, the interpretation of C_s would be difficult. In this paper, it is not attempted to calculate partial factors too. Consequently, the formulation can be simple for now. In the future, it may be extended.

5 Solution for many load combinations

As far as the author knows, an analytical solution of $\delta(d, n)$ has not been found. In this section an approximation is derived for large numbers of points n. The derivation consists of three steps. First the number of facets F of the polytope is approximated from the number of vertices n. Second, the circum-radius ρ of the facets is approximated. Third, the circum-radius Re of the polytope is derived.

If the points of a polytope are uniformly distributed on a sphere the expected number of facets F can be calculated from [9]

$$F = \frac{2}{d} \frac{\gamma((d-1)^2)}{\gamma(d-1)^{d-1}} (n + O(1)) \tag{8}$$

where function γ is defined recursively by

$$\begin{aligned} \gamma(0) &= \tfrac{1}{2} \\ \gamma(p) &= \frac{1}{2\pi p\, \gamma(p-1)} \end{aligned} \tag{9}$$

Constant value O(1) can be neglected for large values of n.

A uniform distribution of points is clearly not the optimal distribution with respect to deviations from the sphere. However, it is assumed that the optimal number of facets will not be much different. After all, for a polygon on a circle even for small numbers of vertices the number of facets is the same for both randomly generated points and optimally distributed points.

Consider a sphere in d-space. Its surface A area is [10],

$$A = \frac{\pi^{\frac{d}{2}}}{\frac{d}{2}!} d R^{d-1} \tag{10}$$

where R is the radius of the sphere. Consider a simplex in d-space. Its surface area is [11]

$$A = \sqrt{\frac{d+1}{2^d}} \frac{a^d}{d!} \tag{11}$$

where a is the length of each side. This area can also be expressed in the radius ρ of its circum-sphere.

$$A = \left(\frac{d+1}{d}\right)^{\frac{d}{2}} \frac{\sqrt{d+1}}{d!} \rho^d \tag{12}$$

When d = 2 the simplex reduces to an equilateral triangle with area $A = \frac{3}{4}\sqrt{3}\rho^2$.

Suppose that a d-sphere is completely covered by (d - 1)-simplices. The number of simplices is F. The simplices are approximately of the same size and form a convex envelope tightly around the sphere (Fig. 8).

The total surface area of the simplices is approximately equal to the area of the sphere. Using Eq. 10 and 12 we obtain

$$F\left(\frac{d}{d-1}\right)^{\frac{d-1}{2}} \frac{\sqrt{d}}{(d-1)!}\ \rho^{d-1} \approx \frac{\pi^{\frac{d}{2}}}{\frac{d}{2}!} d\ Ri^{\,d-1}\ . \tag{13}$$

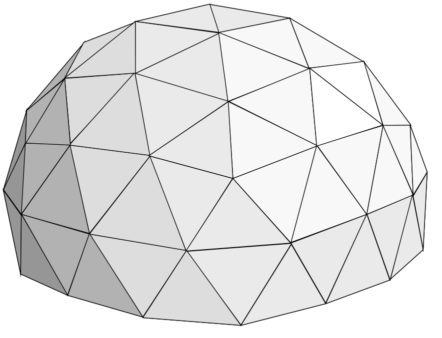

Figure 8. Triangles covering a sphere part in three-dimensional space (geodetic dome)

This approximation becomes better when the number of simplices is increased. The latter equation can be rewritten as

$$\left(\frac{\rho}{Ri}\right)^{d-1} \approx \frac{1}{F}\frac{d!}{\frac{d}{2}!}\left(\frac{\pi}{d}\right)^{\frac{d}{2}}(d-1)^{\frac{d-1}{2}} \tag{14}$$

Consider a sphere with radius *Re*.

$$Re^2 = \sum_{i=1}^{d} x_i^2 \tag{15}$$

where x_i are the space coordinates. The sphere is cut by a hyper plane at a distance *Ri* of the origin.

$$Re^2 = x_1^2 + \sum_{i=2}^{d} x_i^2 = Ri^2 + \rho^2 \tag{16}$$

This can be written as

$$\delta^2 = \frac{Re^2}{Ri^2} = 1 + \frac{\rho^2}{Ri^2} \tag{17}$$

Substitution of Eq. 8 and 14 into Eq. 17 gives

$$\delta \approx \sqrt{1+\frac{a}{(n-b)^{\frac{2}{d-1}}}} \tag{18}$$

where

$$a = \left(\frac{d!}{\left(\frac{d}{2}-1\right)!} \left(\frac{\pi}{d}\right)^{\frac{d}{2}} (d-1)^{\frac{d-1}{2}} \frac{\gamma(d-1)^{d-1}}{\gamma((d-1)^2)} \right)^{\frac{2}{d-1}} \tag{19}$$

and

$$b = \mathrm{O}(1) \tag{20}$$

Eq. 18 shows how δ depends on n for large n. However, in the previous derivation a full sphere is considered and we are only interested in the part of space for which values of are positive. Therefore, the number of points n needs to be reduced to $n/2^d$ and boundary points need to be added. These corrections do not change Eq. 18 only the values of the coefficients a and b are changed.

Substitution of Eq. 18 in Eq. 5 gives [1]

$$0 = \frac{-a}{\delta_{\text{opt}}(d-1)(n_{\text{opt}}-b)^{\frac{d+1}{d-1}}} C_s + C_a \tag{21}$$

from which can be solved.

$$\boxed{n_{opt} = b + \left(\frac{a}{\delta_{opt}(d-1)} \frac{C_s}{C_a} \right)^{\frac{d-1}{d+1}}} \tag{22}$$

where

$$\delta_{\text{opt}} = \sqrt{1+\frac{a}{\left(n_{\text{opt}}-b\right)^{\frac{2}{d-1}}}}\ . \tag{23}$$

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[1] In personal emails, Prof. I. Bárány (University College London) suggested the below approximation of δ for large n. In this, c is a small positive constant. Unfortunately, this relation is too complicated for solving the optimum number of load combinations.

$$\delta \approx \frac{1}{1-\left(\frac{c \log n}{n}\right)^{\frac{2}{d-1}}}$$

6 Approximation formula

In this section the coefficients a and b of Eq. 18 are determined. The approach is to fit Eq. 18 as well as possible to the solutions for few load combinations.

A particular polytope has the following point co-ordinates.

$$\begin{aligned}
&(1,0,0,0\,...,0)\\
&\left(\tfrac{1}{2}\sqrt{2},\tfrac{1}{2}\sqrt{2},0,0\,...,0\right)\\
&\left(\tfrac{1}{3}\sqrt{3},\tfrac{1}{3}\sqrt{3},\tfrac{1}{3}\sqrt{3},0\,...,0\right)\\
&...\\
&\left(\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d}\,...,\tfrac{1}{d}\sqrt{d}\right)
\end{aligned} \tag{24}$$

The numbers can have any position between the brackets, so also (0,1,0,0 …,0), (0,0,1,0 …,0), (0,0,0,1 …,0) and (0,0,0,0 …,1) are points of the polytope. The total number of points of this polytope is $2^d - 1$. The ratio Re/Ri is (See appendix)

$$\frac{Re}{Ri} = \sqrt{\sum_{i=1}^{d}\left(\sqrt{i}-\sqrt{i-1}\right)^2} \tag{25}$$

It is expected that this is an optimal polytope in the sense that the point co-ordinates are such that Re/Ri has the smallest value. This is suggested by the fact all facets have the same shape and the same distance to the origin. This has been confirmed for d = 2, 3, 4, 5 and 6.

The constant b is only important for few load combinations. Selected is $b = d - 1$. With this choice Eq. 18 gives realistic results for small d and n. The constant a is determined by matching Eq. 18 to the solution of the polytope of this section,

$$\sqrt{1+\frac{a}{(2^d-d)^{\frac{2}{d-1}}}} = \sqrt{\sum_{i=1}^{d}\left(\sqrt{i}-\sqrt{i-1}\right)^2} \tag{26}$$

from which a can be solved.

$$a = \left(\sum_{i=1}^{d}\left(\sqrt{i}-\sqrt{i-1}\right)^2 - 1\right)\left(2^d-d\right)^{\frac{2}{d-1}} \tag{27}$$

Table 3 contains values of a. For $d < 13$ coefficient a can be approximated by

$$a = 3.37 - \frac{7.54}{d+0.807}\,. \tag{28}$$

Figure 9 shows the approximation formula (line) and the exact values (dots). For small n the deviations are considerable. For $n > 2^d$ there is a good agreement. In Section 7 it is shown that n is usually very large.

Table 3. Values of coefficient a as a function of the number of dimensions d

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a	0	0.69	1.4	1.8	2.1	2.3	2.4	2.5	2.6	2.7	2.7	2.8	2.9	2.9	3.0

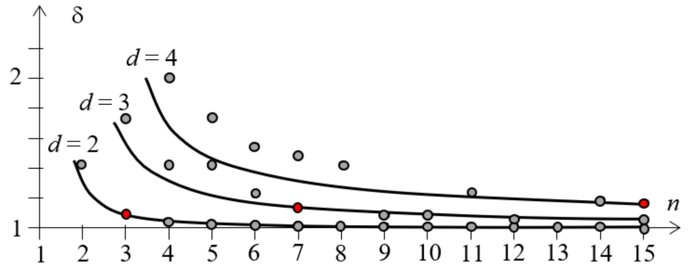

Figure 9. Curve fit of δ for d = 2, d = 3 and d = 4

The structural cost C_s is for a design optimised in a number of cycles and evaluated by accurate probability analyses (reliability based optimisation). This is hardly ever done, of course. Therefore, it is convenient to introduce the cost of a similar previous project.

$$C_s' = \delta_{\text{pre}} C_s \tag{29}$$

where

$$\delta_{\text{pre}} \approx \sqrt{1 + \frac{a}{\left(n_{\text{pre}} - b\right)^{\frac{2}{d-1}}}} \tag{30}$$

7 Examples

A *high-rise building* will cost approximately 10^6 euros of which 20% is for the steel structure. In design 10 independent load cases need to be considered including dead load, live load in varying proportions on the floors, wind loads from several directions (mutually exclusive) and accidental loads, therefore, $d = 10$. An extra structural analysis (geometrically non-linear) takes 1 minute of computation time and costs approximately 3 euros. We approximate $\delta_{pre} = \delta_{opt} = 1.0$. The coefficients are $a = 2.7$ and $b = 9$. The optimal number of load combinations is approximately (Eq. 22)

$$n_{opt} = 9 + \left(\frac{2.7}{1.0 \times 1.0 \times 9} \frac{0.2 \times 10^6}{3} \right)^{\frac{9}{11}} = 3300$$

The analyses would take two days of computation time.

A *storm surge barrier* costs approximately 2×10^7 euros. Water pressure and ship impact need to be considered in design of the steel structure. Self-weight can be calculated accurately and requires no partial factor. Consequently, only two load cases have partial factors and $d = 2$. Each finite element analysis takes approximately two hours and costs 700 euros. The optimal number of load combinations is (Eq. 7)

$$n_{opt} = 1 + \sqrt[3]{\frac{\pi^2}{16} \frac{2 \times 20^7}{700}} = 27 \ .$$

This means that instead of one collision with a large ship rather 27 different collisions should be analysed each with a somewhat smaller ship in order to obtain the smallest project costs. Equation 2 can be used to show that the largest of the somewhat smaller ships is 8% smaller than for 3 load combinations.

An *offshore platform* will cost approximately 10^8 euros. The number of load cases that need to be considered in design is 8. The cost of analysing one load combination is 100 euros. The structural cost is based on a previously constructed platform that has been designed with 16 load combinations. The constants are $a = 2.5$ and $b = 7$. The previous margin δ is (Eq. 18)

$$\delta_{pre} = \sqrt{1 + \frac{2.5}{(16-7)^{\frac{2}{7}}}} = 1.5 \ .$$

We estimate the optimum margin to be $\delta_{opt} = 1.1$. The optimum number of load combinations is approximately (Eq. 22)

$$n_{opt} = 7 + \left(\frac{2.5}{1.5 \times 1.1 \times 7} \frac{10^8}{100} \right)^{\frac{7}{9}} = 14000 \ .$$

Therefore, the optimum margin is

$$\delta_{opt} = \sqrt{1 + \frac{2.5}{(14000-7)^{\frac{2}{7}}}} = 1.08\,.$$

The reduction of the structure cost compared to the previous project is

$$\frac{1.5-1.08}{1.5}10^8 - (14000-16)\,100 = 0.27 \times 10^8\,,$$

which is 27%.

8 Discussion

This paper estimates the number of load combinations that should be used in design. It does not give the values of the partial factors that need to be used in these optimal load combinations. At present, computing optimal partial factors is only possible for few load cases and few load combinations. For many cases and combinations the computation takes more than a week on a modern computer, which is impractical. Without optimal partial factors, the theory in this paper cannot be used in practice.

The value of the failure probability is represented by the diameter of the in-sphere. This value has no influence on the optimal number of load combinations. In addition, it does not matter whether the failure probability is an *annual maximum value* to protect individual citizens or a *design live target value* to protect society and investments.

A parametric structural design can be optimised such that the costs, including failure costs, are smallest, while the failure probability is acceptable. In this method, there is no need for load combinations or partial factors. This computation requires many probabilistic analyses which takes much time. The examples in this paper show that when this optimisation becomes available in practice, substantial material savings can be obtained.

Nevertheless, structural analysis according to codes of practice with partial factors and load combinations is a clever system, which is difficult to improve upon. It is definitely appropriate to credit the generation of engineers that developed this successful system.

9 Conclusion

A geometrical model is made of load combinations used in structural analysis including limit state function and reliability requirement. It is shown that the load combinations can be optimised to obtain the least cost without compromising the required reliability. From this concept, a formula is derived for the optimum number of load combinations.

$$n_{\text{opt}} = b + \left(\frac{a}{\delta_{\text{pre}} \delta_{\text{opt}} (d-1)} \frac{C_s'}{C_a} \right)^{\frac{d-1}{d+1}}$$

where d is the number of load cases, C_s' is the cost of the structure and C_a is the cost of analysing one extra load combination. The coefficients are $a = 3.37 - \frac{7.54}{d + 0.807}$, $b = d - 1$. The margins δ_{pre} and δ_{opt} are

$$\delta_{\text{pre}} = \sqrt{1 + \frac{a}{\left(n_{\text{pre}} - b\right)^{\frac{2}{d-1}}}}$$

$$\delta_{\text{opt}} = \sqrt{1 + \frac{a}{\left(n_{\text{opt}} - b\right)^{\frac{2}{d-1}}}}$$

where n_{pre} is the number of load combinations used in the previous similar structure from which the structural costs C_s' are derived.

Examples show that the optimal number of combinations can be much larger than used in practice.

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Appendix

In this appendix, equation 25 is derived. The equation gives the ratio of the circum-sphere *Re* over in-sphere *Ri* of the polytope in Chapter 6. The circum-sphere has radius 1. All points are on this sphere. This can be verified by calculating the distances of the points to the origin. For example:

$$\sqrt{\left(\tfrac{1}{3}\sqrt{3}\right)^2+\left(\tfrac{1}{3}\sqrt{3}\right)^2+\left(\tfrac{1}{3}\sqrt{3}\right)^2+0\ldots+0}=1$$

A plane goes through *d* points. These points are

$$(1,0,0,0\ldots,0)$$
$$\left(\tfrac{1}{2}\sqrt{2},\tfrac{1}{2}\sqrt{2},0,0\ldots,0\right)$$
$$\left(\tfrac{1}{3}\sqrt{3},\tfrac{1}{3}\sqrt{3},\tfrac{1}{3}\sqrt{3},0\ldots,0\right)$$
$$\ldots$$
$$\left(\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d},\tfrac{1}{d}\sqrt{d}\ldots,\tfrac{1}{d}\sqrt{d}\right)$$

The function of this plane is

$$x_1+\left(\sqrt{2}-1\right)x_2+\left(\sqrt{3}-\sqrt{2}\right)x_3\ldots+\left(\sqrt{d}-\sqrt{d-1}\right)x_d=1\,.$$

This can be verified by substituting any point into the function. For example,

$$\tfrac{1}{2}\sqrt{2}+\left(\sqrt{2}-1\right)\tfrac{1}{2}\sqrt{2}+\left(\sqrt{3}-\sqrt{2}\right)0\ldots+\left(\sqrt{d}-\sqrt{d-1}\right)0=1$$

In general [10], a point (x_o,y_o,z_o) has a distance *Ri* to plane $Ax+By+Cz+D=0$, of

$$Ri=\frac{|Ax_o+By_o+Cz_o+D|}{\sqrt{A^2+B^2+C^2}}\,.$$

Therefore, the distance of the origin to the present plane is

$$Ri=\frac{1}{\sqrt{1+\left(\sqrt{2}-1\right)^2+\left(\sqrt{3}-\sqrt{2}\right)^2\ldots+\left(\sqrt{d}-\sqrt{d-1}\right)^2}}\,.$$

This can be written as

$$Ri=\frac{1}{\sqrt{\sum_{i=1}^{d}\left(\sqrt{i}-\sqrt{i-1}\right)^2}}\,.$$

Since *Re* = 1,

$$\frac{Re}{Ri}=\sqrt{\sum_{i=1}^{d}\left(\sqrt{i}-\sqrt{i-1}\right)^2}$$

Q.E.D.

The perpendicular projection of the origin onto the plane could have negative coordinates. If it were, the in-sphere is not constrained by the plane but by an edge. It has been verified that this is not the case.

The polytope is the convex hull of $n = d + d\,(d\text{-}1)/2 + d\,(d\text{-}1)(d\text{-}2)/(2 \times 3) \ldots + 1 = 2^d - 1$ points. For $d = 2$ and $d = 3$ the shape has been visually inspected. For $d = 4$ and higher this is obviously not possible. The author is aware that the proof in this appendix lacks mathematical rigor. Mathematicians are kindly invited to do a better job than me.