Stability of Baker Trusses

Lazlo W. Bleker
Delft University of Technology, the Netherlands
Skidmore, Owings & Merrill, Chicago, USA

Baker Trusses have the ability to be more efficient in material usage than conventional truss types and are thus of interest to be considered by engineers. It is known that under certain parameters and load conditions the web-nodes of a Baker Truss can become unstable in the out-of-plane direction. Up to now the way in which for this failure mechanism is checked in practice, is by calculating the member forces and the geometric stiffness of each web-node. In this paper, two equations are given; one for a top-loaded and one for a bottom-loaded Baker Truss, allowing for direct calculation of the web-nodal geometric stiffness without the need to calculate the member forces. These equations show that most web-nodes for the top-loaded condition are unstable and that most web-nodes for the bottom-loaded condition are stable.

Key words: Baker Truss, geometric stiffness, stability

1 Introduction

The Baker Truss (Figure 1), also known as the discrete optimal truss, is a type of truss first conceived by William Baker. Linear elastic analysis shows that for many depth-to-span ratios and load-patterns, the Baker Truss is much more efficient (i.e. uses less material to achieve equal stiffness or strength) than conventional truss types [Baker, 2013]. It is therefore of interest to use this truss to save on material and cost, but also to create new meaningful architecture [SOM, 2020 (a); SOM 2020 (b)].

A major drawback of the Baker Truss is that under certain conditions (e.g. span-to-depth ratio or load pattern) the web-nodes can become unstable. In this instability web-nodes displace in the out-of-plane direction under the influence of in-plane loading. The stability of such a node is governed by the sign of the geometric stiffness (Annex A). This paper presents the generic expression for the geometric stiffness of the web-nodes of a uniformly top-loaded Baker Truss and a similar expression for a uniformly bottom-loaded Baker Truss.

2 Method

Figure 1 shows the truss which is to be analysed including the parameters used. Dimensions L and h denote the total span and height of the truss respectively. Variable b is the total number of bays (panels) per halve span. The web-nodes are located at half height and in horizontal direction at three quarters width of the bays. For the example truss in Figure 1, b is equal to 4. n denotes the bay number, starting at 1 on either support and increasing by 1 for every bay towards the middle of the truss. Finally, with F denoting the downwards directed point-loads at all upper nodes of the truss, as shown in the figure, the problem is uniquely defined.

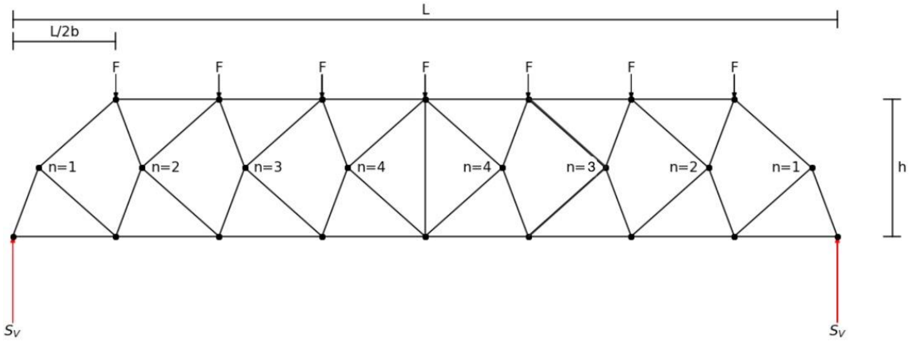


Figure 1: Parameters of the Baker Truss

The top and bottom chord nodes are fixed in out-of-plane direction e.g. by the chords being continuous or by floor slabs at the height of the chords. Therefore, only the web nodes can buckle in the out-of-plane direction. The geometrical stiffness is given by equation (1). For a derivation of this equation see Annex A.

$$k_n = \sum_{i=1}^{j} \frac{N_i}{L_i} \qquad (1)$$

in which

k_n is the geometric stiffness of node n;

j is the number of nodes connected to node n;

L_i is the length of the member connecting node n to node i;

N_i is the normal force of the member connecting node n to node i;

The interpretation of the geometric stiffness is that a node that is moved perpendicular to the plane of the truss will push back with a force. The magnitude of this force is the absolute value of the displacement times the geometrical stiffness k_n. Consequently, a node is stable if its geometrical stiffness is positive.

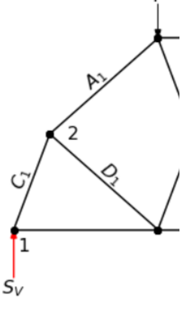
A Baker truss is statically determinate and therefore all member forces can be determined using only equilibrium equations. The derivation is based upon analytically derived member forces being substituted in equation (1) to obtain the desired equation.

3 Derivation

This section presents the full derivation of the expression for the geometric stiffness of any web-node in a top-loaded Baker Truss. The derivation of the bottom-loaded truss only requires changes in a few equations of the top-loaded truss derivation and is not included. The derivation can be divided into five consecutive steps:

1. Deriving the support reactions, followed by the forces in web members of bay 1 (leftmost bay).
2. Deriving the expression relating the forces in web members of bay n to bay n – 1.
3. Decoupling the previous expression such that each relevant member of bay n individually relates to the same member of bay n – 1.
4. Converting the expressions relating forces in bay n to the forces in bay n – 1 (i.e. a series) to an equation containing n as a variable.
5. Determining all member forces and substituting these into the general expression for nodal geometric stiffness.

The first step involves calculating the support reaction S_V, as well as the member forces A_1, C_1 and D_1 shown in the figure below.



$$S_V = -F\left(\frac{2b-1}{2}\right) = -F\left(b - \frac{1}{2}\right) \quad (2)$$

$$\sum F_{V,1} = 0 = -\frac{hC_1}{2L_1} + S_V = -\frac{hC_1}{2L_1} - F\left(b - \frac{1}{2}\right) \quad (3)$$

$$\sum F_{V,2} = 0 = -\frac{hA_1}{2L_2} - \frac{hC_1}{2L_1} + \frac{hD_1}{2L_2} = -\frac{A_1}{L_2} + \frac{C_1}{L_1} + \frac{D_1}{L_2} \quad (4)$$

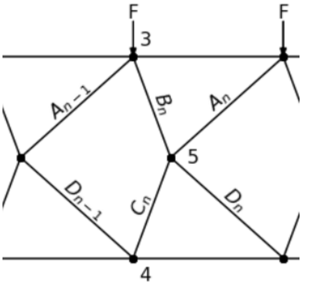
$$\sum F_{H,2} = 0 = \frac{3LA_1}{8bL_2} - \frac{LC_1}{8bL_1} + \frac{3LD_1}{8bL_2} = \frac{3A_1}{L_2} - \frac{C_1}{L_1} + \frac{3D_1}{L_2} \quad (5)$$

In which L, L_1, L_2 and b denote the total span length, the length of the shorter web members, the length of the longer web members and the total number of bays per half span respectively. Four equations and four unknowns means a solution for the unknowns can be found:

$$A_1 = \frac{F}{h}\left(\frac{2L_2}{3} - \frac{4bL_2}{3}\right) \quad (6)$$

$$D_1 = \frac{F}{h}\left(\frac{2bL_2}{3} - \frac{L_2}{3}\right) \quad (7)$$

The next step is to relate A_n and D_n to A_{n-1} and D_{n-1}. The figure below shows the naming of relevant members and nodes.



$$\sum F_{V,3} = 0 = \frac{hA_{n-1}}{2L_2} + \frac{hB_n}{2L_1} + F \quad (8)$$

$$\sum F_{V,4} = 0 = -\frac{hD_{n-1}}{2L_2} - \frac{hC_n}{2L_1} \quad (9)$$

$$\sum F_{V,5} = 0 = -\frac{hA_n}{2L_2} - \frac{hB_n}{2L_1} + \frac{hC_n}{2L_1} + \frac{hD_n}{2L_2} = -\frac{A_n}{L_2} - \frac{B_n}{L_1} + \frac{C_n}{L_1} + \frac{D_n}{L_2} \quad (10)$$

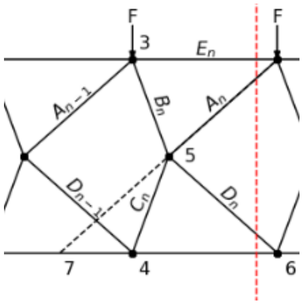
$$\sum F_{H,5} = 0 = \frac{3LA_n}{8bL_2} - \frac{LB_n}{8bL_1} - \frac{LC_n}{8bL_1} + \frac{3LD_n}{8bL_2} = \frac{3A_n}{L_2} - \frac{B_n}{L_1} - \frac{C_n}{L_1} + \frac{3D_n}{L_2} \quad (11)$$

The above system of four equations and six unknowns can be reduced to a system of 2 equations and four unknowns in which B_n and C_n have been eliminated.

$$D_n = -\frac{2A_{n-1}}{3} - \frac{4FL_2}{3h} + \frac{D_{n-1}}{3} \quad (12)$$

$$A_n = \frac{A_{n-1}}{3} + \frac{2FL_2}{3h} - \frac{2D_{n-1}}{3} \quad (13)$$

For the third step the equations (12) and (13) are to be decoupled such that in the expression for D_n, D_n is only related to D_{n-1} and that in the expression for A_n, A_n is only related to A_{n-1}. An expression relating A_{n-1} to D_{n-1} could accomplish this goal, and it is derived by creating a cut in bay n as shown in the figure below.



$$\sum M_6 = 0 = -xA_n - hE_n + \frac{nL}{4b}F(n-1) + \frac{nL}{2b}S_V = -xA_n - hE_n + F\left(\frac{Ln^2}{4b} - \frac{nL}{2}\right) \tag{14}$$

$$\sum M_7 = 0 = -xD_n - hE_n + \left(\frac{nL}{4b} - \frac{3L}{4b}\right)F(n-1) + \left(\frac{nL}{2b} - \frac{3L}{4b}\right)S_V =$$

$$-xD_n - hE_n + F\left(\frac{Ln^2}{4b} - \frac{nL}{2} - \frac{3nL}{4b} + \frac{3L}{4} + \frac{3L}{8b}\right) \tag{15}$$

In the above equations x denotes the orthogonal distance from the force in member A_n to node 6, which is equal to the orthogonal distance from the force in member D_n to node 7. Equations (14) and (15) can be combined into one equation in which E_n has been eliminated and thus the only unknowns left are A_n and D_n.

$$D_n = A_n + \frac{F}{x}\left(-\frac{3nL}{4b} + \frac{3L}{4} + \frac{3L}{8b}\right) \tag{16}$$

When n is substituted by n-1 the sought-after relation is found.

$$D_{n-1} = A_{n-1} + \frac{F}{x}\left(-\frac{3nL}{4b} + \frac{3L}{4} + \frac{9L}{8b}\right) \tag{17}$$

Combining equations (17) and (12) as well as (17) and (13) yields the desired pair of decoupled series.

$$A_n = c_1 A_{n-1} + c_2 n + c_3$$

$$c_1 = -\frac{1}{3} \qquad c_2 = \frac{FL}{2bx} \qquad c_3 = \frac{2FL_2}{3h} - \frac{FL}{2x} - \frac{3FL}{4bx} \tag{18}$$

$$D_n = c_4 D_{n-1} + c_5 n + c_6$$

$$c_4 = -\frac{1}{3} \qquad c_5 = -\frac{FL}{2bx} \qquad c_6 = -\frac{4FL_2}{3h} + \frac{FL}{2x} + \frac{3FL}{4bx} \tag{19}$$

The fourth step is to convert the derived series into an equation relating A_n to just A_1 and n, as well as for D_n. This is done by firstly expanding equation (18) into a summation of series.

$$A_n = c_1 A_{n-1} + c_2 n + c_3 = c_1^{n-1} A_1 + c_2\big((c_1^{n-2} + c_1^{n-3} + \cdots + c_1^1 + c_1^0) + (c_1^{n-3} + c_1^{n-4} + \cdots + c_1^1 + c_1^0) + \cdots + (c_1^1 + c_1^0) + (c_1^0)\big) + c_2(c_1^{n-2} + c_1^{n-3} + \cdots + c_1^1 + c_1^0) + c_3(c_1^{n-2} + c_1^{n-3} + \cdots + c_1^1 + c_1^0) \tag{20}$$

Making use of the known relation (21) [Stewart, 2011], equation (20) can be rewritten as equation (22). In an identical manner an equation for D_n can be derived.

$$x^{n-1} + x^{n-2} + \cdots + x^1 + x^0 = \frac{x^n - 1}{x-1} \qquad x \neq 1 \quad \text{and} \quad n \geq 1 \tag{21}$$

$$A_n = A_1 c_1^{n-1} + \frac{c_2}{c_1 - 1}\left(\frac{c_1^n - 1}{c_1 - 1} + c_1^{n-1} - n - 1\right) + c_3\left(\frac{c_1^{n-1} - 1}{c_1 - 1}\right) \tag{22}$$

$$D_n = D_1 c_4^{n-1} + \frac{c_5}{c_4 - 1}\left(\frac{c_4^n - 1}{c_4 - 1} + c_4^{n-1} - n - 1\right) + c_6\left(\frac{c_4^{n-1} - 1}{c_4 - 1}\right) \tag{23}$$

For the final step, the total geometric stiffness k_n will be split up in two parts: the contribution by the normal force in members A_n and D_n, k_a, and the contribution of members B_n and C_n, k_b. A simple expression for k_a can be found by substituting equations (22) and (23) into equation (24) followed by substituting equations (6) and (7) as well as all constants of equations (18) and (19).

$$k_a = \frac{A_n}{L_2} + \frac{D_n}{L_2} = \frac{F}{h}\left(\frac{2b - \frac{5}{2}}{(-3)^n} - \frac{1}{2}\right) \tag{24}$$

Equation (11) can be rewritten in terms of k_1 opening up a simple way of calculating k_2.

$$0 = \frac{3A_n}{L_2} - \frac{B_n}{L_1} - \frac{C_n}{L_1} + \frac{3D_n}{L_2}$$

$$k_b = \frac{B_n}{L_1} + \frac{C_n}{L_1} = 3k_a = \frac{F}{h}\left(\frac{6b - \frac{15}{2}}{(-3)^n} - \frac{3}{2}\right) \tag{25}$$

Thus, the total geometric stiffness k can be calculated:

$$k_n = k_a + k_b = \frac{F}{h}\left(\frac{2b - \frac{5}{2}}{(-3)^n} - \frac{1}{2}\right) + \frac{F}{h}\left(\frac{6b - \frac{15}{2}}{(-3)^n} - \frac{3}{2}\right) = \frac{F}{h}\left(\frac{8b - 10}{(-3)^n} - 2\right) \tag{26}$$

Q.E.D.

4 Results

The geometric stiffness of the web-nodes of a uniformly top-loaded baker truss is given by

$$k_n = \frac{F}{h}\left(\frac{8b - 10}{(-3)^n} - 2\right) \tag{27}$$

The geometric stiffness of the web-nodes of a uniformly bottom-loaded baker truss is given by

$$k_n = \frac{F}{h}\left(\frac{8b + 2}{(-3)^n} + 2\right) \tag{28}$$

Since F and h do not influence the sign of the geometric stiffness in equations (27) and (28), these can be taken out of the equation so that a unit-less measure for the nodal stability is left. Figure 2 shows a plot of this stability measure for each web-node of a top-loaded truss (left) and a bottom-loaded truss (right) for the case $b = 8$.

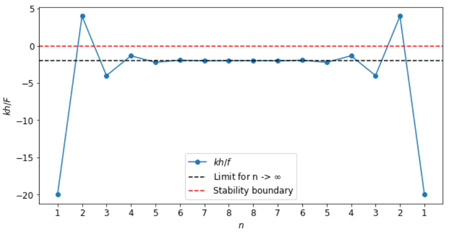


Figure 2a: Stability of a top-loaded truss; most web nodes are unstable.

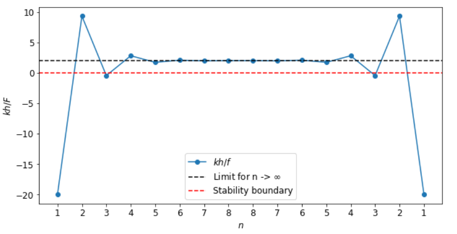


Figure 2b: Stability of a bottom-loaded truss; most web nodes are stable.

The limits that the geometric stiffness approaches as *n* increases, for a constant value of *b*, is for the top- and bottom-loaded truss respectively equal to equation (29) and (30).

$$\lim_{n\to\infty} \frac{F}{h}\left(\frac{8b-10}{(-3)^n} - 2\right) = -2\frac{F}{h} \quad (29)$$

$$\lim_{n\to\infty} \frac{F}{h}\left(\frac{8b+2}{(-3)^n} + 2\right) = 2\frac{F}{h} \quad (30)$$

5 Conclusion

This study shows that the geometric stiffness of the web-nodes of a uniformly top- and bottom-loaded Baker Truss can be calculated using a simple expression, greatly reducing the number of steps involved and giving more insight when compared with the conventional method of first calculating all member forces. The results of the equations have been compared to calculations done by a custom FEM code and are accurate up to many numbers after the decimal point (Annex B). It is shown that, for a positive value of the forces *F*, most nodes of a top-loaded truss are unstable, while most nodes of a bottom-loaded truss are stable. When the forces are directed upwards the opposite is true in which most nodes of a top-loaded truss are stable, and most nodes of a bottom-loaded truss are unstable. It can be concluded that for most Baker Trusses some of the web-nodes need to be supported or stiffened by some moment transferring nodes in the out-of-plane direction in order to guarantee stability. For top-loaded Baker Trusses this is even true for most of the web-nodes.

6 Acknowledgements

The findings in this paper could not have been done without the inspiration and feedback from the people at the SOM Chicago office, in particular Arek Mazurek, for his feedback and encouragement throughout my stay. I would also like to thank Pierre Hoogenboom (Delft University of Technology) for supervising the project, as well as for the valuable feedback on the manuscript he has given.

Literature

Baker, W.F. (2013). Structural innovation: Combining classic theories with new technologies. *SOM*. pp. 18-19.

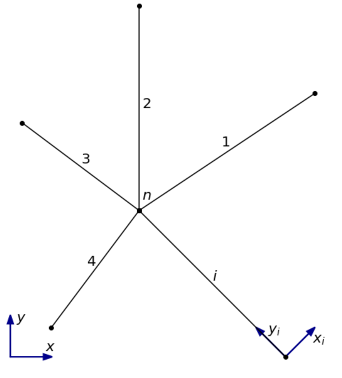
SOM (a). 100 Mount Street, https://www.som.com/projects/100_mount_street, accessed on January 29th 2020

SOM (b). 800 West Fulton Market, https://www.som.com/projects/800_west_fulton_market, accessed on January 29th 2020

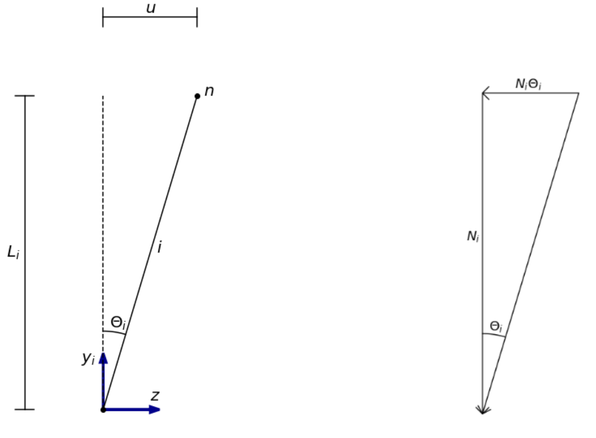
Stewart, J. *Calculus Early Transcendentals 7th edition*, Brooks/Cole, 2011

Annex A. Derivation of the geometric stiffness equation

This annex presents the derivation of the general expression for the geometric stiffness of a node in a truss. The working axis of the stiffness is perpendicular to the plane of the truss. The assumption is made that all nodes connecting to the node of which the geometric stiffness is to be determined are fixed in the out of plane direction. Consider a node *n* connected by one member per node to *j* nodes.



Node *n* is displaced by *u* in the *z*-direction (out-of-plane). Because all nodes connected to *n* are fixed in the out-of-plane direction, each member *i* with a length L_i is rotated by an angle $\theta_i = \frac{u}{L_i}$. The force exerted on node *n* by the normal force in member *i*, in the deformed state, is split up in a component in *z*-direction and a component in y_i-direction.



The component of the force in y_i-direction in the undeformed state is equal to the normal force in member *i*, N_i, since in this case the member is aligned with the y_i-axis. In the deformed state the component of the force in y_i-direction exerted on node *n* must remain equal to the undeformed state in order to preserve nodal equilibrium in the x,y-plane. Assuming that u, and by extension θ_i, is small, the component of the force in z-direction can be calculated.

$$F_{i,z} = -\theta_i N_i = -u \frac{N_i}{L_i} \quad (1)$$

The general definition of stiffness is used in which a positive stiffness corresponds to a force in the opposite direction of the displacement. The force is taken as the sum of forces in z-direction of all members connected to node n.

$$k_n = -\frac{1}{u}\sum_{i=1}^{j} F_{i,z} = -\frac{1}{u}\sum_{i=1}^{j} -u\frac{N_i}{L_i} = \sum_{i=1}^{j} \frac{N_i}{L_i} \quad (2)$$

Q.E.D.

Annex B. Verification

Table 1: Difference between FEM-calculated k_n and equation-calculated k_n for several Baker Trusses

Load type	*F [kN]*	*h [m]*	*b*	*n*	FEM k_n [kN/m]	Equation k_n [kN/m]	Difference
Top	10	5	2	1	-8.00000	-8.00000	0.00000%
Top	10	5	2	2	-2.66667	-2.66667	0.00000%
Top	15	7	4	1	-20.0000	-20.0000	0.00000%
Top	15	7	4	2	0.952381	0.952381	0.00000%
Top	15	7	4	3	-6.03175	-6.03175	0.00000%
Top	15	7	4	4	-3.70370	-3.70370	0.00000%
Bottom	12	8	6	1	-22.0000	-22.0000	0.00000%
Bottom	12	8	6	2	11.33333	11.33333	0.00000%
Bottom	12	8	6	3	0.222222	0.222222	0.00000%
Bottom	12	8	6	4	3.925925	3.925925	0.00000%
Bottom	12	8	6	5	2.691358	2.691358	0.00000%
Bottom	12	8	6	6	3.102881	3.102881	0.00000%