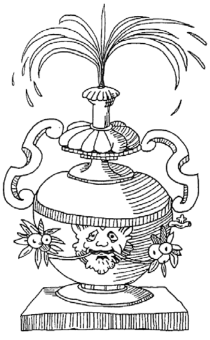
Heron's fountain 19



Finds and ideas with a surprising element similar to the playful inventions of Heron of Alexandria, after whom this journal is named

The formula of Heron

The formula of Heron expresses the area A of a triangle in the lengths a, b, c of its sides.

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where s is the semiperimeter of the triangle defined by

$$s = \frac{a+b+c}{2} \tag{2}$$

Substitution of the semiperimeter into the formula leads to an equivalent expression.

$$A = \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \tag{3}$$

Though two millennia old, this formula is practical because measuring lengths is often easier than measuring angles. In what follows, a modern proof is given. It uses the cosine rule, which surely many readers remember from secondary school.

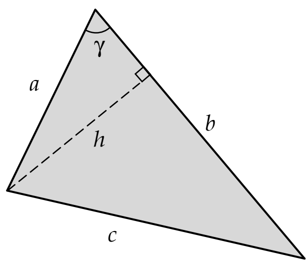


Figure 1. Heron's formula expresses the area of a triangle in the lengths of its sides.

Trigonometric proof of Heron's formula

According to the cosine rule (fig. 1)

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{4}$$

The depth of the triangle is

$$h = a\sin\gamma \tag{5}$$

and its area is

$$A = \frac{1}{2} b h \tag{6}$$

Equation 5 is substituted into 6 and equation 4 is substituted into the result

$$A = \frac{1}{2} ba \sin\gamma = \frac{1}{2} ba\sqrt{1-\cos^2\gamma} = \frac{1}{2} ba\sqrt{1-(\frac{a^2+b^2-c^2}{2ab})^2} = \frac{1}{2} ba\sqrt{\frac{(2ab)^2-(a^2+b^2-c^2)^2}{(2ab)^2}} =$$

$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)^2} = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

Brahmagupta's formula; a generalization of Heron's formula

A circle can be drawn through the three vertices of a triangle. This is called a cyclic triangle. When a second triangle is inserted in the circle having one side in common with the first one, a cyclic quadrilateral is obtained (fig. 2). This procedure can be continued to generate more complicated cyclic polygons. We will, however, only discuss the simplest case as an example. The area of the cyclic quadrilateral is the summation of the areas of the two triangles. This problem was first studied by Brahmagupta (598-665). The area A of a cyclic quadrilateral is

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{9}$$

where s is the semiperimeter of the quadrilateral

$$s = \frac{a+b+c+d}{2} \tag{10}$$

It is not difficult to see that for $d = 0$ the quadrilateral degenerates into a single triangle and that Heron's formula is regained.

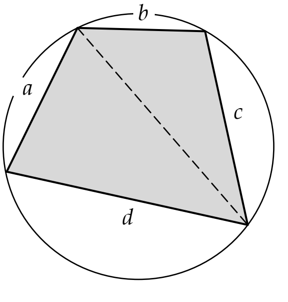


Figure 2. Brahmagupta's approach to determining the area of a cyclic quadrilateral

It needs mentioning that Qin Jiushao arrived independently at the same result in his "mathematical treatise in nine sections" from 1247. Today, Qin is recognised as one of China's most prominent mathematicians.

Building on the foundation of Heron's case

In 2012, Albrecht Hess describes how Brahmagupta's formula follows from Heron's formula by subtraction of two similar triangles (A. Hess, A highway from Heron to Brahmagupta, *Forum Geometricorum*, 12: 191-192). He stated: "*Although I have searched extensively ... , the following derivation of the area of a cyclic quadrilateral from Heron's formula seems to be unknown*". His proof will not be reproduced here, despite it is on secondary school level. For further reading we refer to Wikipedia, which provides extensive information on the formulas of Heron and Brahmagupta, on Qin Jiushao's work and on Hess' contribution.

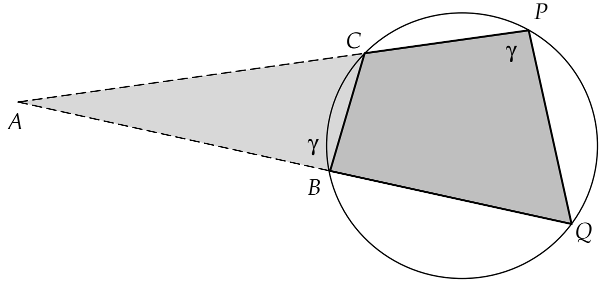


Figure 3. A quadrilateral BCPQ, inscribed in a circle, is extended to yield two triangles, ΔABC and ΔAPQ, which are of similar shape (triangles have three similar angles), but have different sizes.

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