Note on stresses in a chimney due to wind

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Donnell published in 1933 an elegant and rather simple differential equation for the stress analysis in circular cylindrical shells. The civil engineering structural mechanics group in Delft has a long and strong experience in the fifties and sixties of last century in working with this shell equation in close cooperation with the then building institute of TNO. Shallow roof shells, tanks and chimneys were examined. It is known that the equation has restricted reliability for shells in which wavelengths in circumferential direction are relatively large. In 1959 Morley replaced Donnell's equation by a slightly extended equation which retained the essential simplicity of the original but the accuracy does not decrease as the wave length of circumferential distortion increases. The impact on longitudinal bending stresses at the base of chimneys due to wind is not negligible as appears from here summarized research. Application of Morley's theory is easily extended to ring-stiffened and elastically supported chimneys. This note demonstrates how subsequent research in time of individual persons provides stepping stones in the process of discovering.

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1 Difference between Donnell and Morley equations

Consider a circular cylindrical chimney of height l, radius r and wall thickness t as shown in Figure 1. Thickness t is supposed to be small with respect to the radius a. We will consider the chimney under wind load. It is convenient to choose an x-axis in longitudinal direction of this shell with its origin at the base of the chimney, as shown in Figure 1, and a y-axis in circumferential direction. Alternately we will apply the angle $\theta = y/a$ instead of y. A z-axis is normal to the shell surface and outward positive. Displacements in the direction of x, θ and z are u, v and w respectively. The modulus of elasticity is E and Poisson's ratio ν.

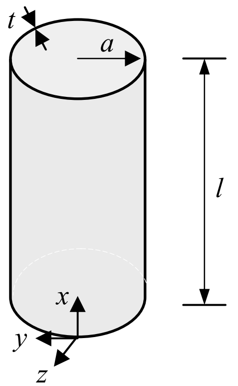

Figure 1. Definition of axes

Donnell equation

At the time Donnell published his equation the extensive theory of Love [1] was at disposal since the end of the nineteenth century and the rigorous theory of Flügge [2] was published in 1934. Donnell [3] succeeded to simplify the theory to the equation for w

$$\Delta\Delta\Delta\Delta\, w + 4K^4 \frac{\partial^4 w}{\partial x^4} = \frac{1}{D}\Delta\Delta\, p \tag{1}$$

where p is the wind pressure in z-direction and Δ the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \theta^2} \;\; ; \quad K^4 = \frac{3(1-\nu^2)}{a^2 t^2} \;\; ; \quad D = \frac{E t^3}{12(1-\nu^2)} \tag{2}$$

Once the displacement w has been solved, the displacements u and v can be found from the equations

$$\begin{aligned} \Delta\Delta u &= -v\frac{1}{a} w_{,xxx} + \frac{1}{a^3} w_{,x\theta\theta} \\ \Delta\Delta v &= -(2+v)\frac{1}{a^2} w_{,xx\theta} - \frac{1}{a^4} w_{,\theta\theta\theta} \end{aligned} \tag{3}$$

The Donnell equation (1) has a few drawbacks. First, it does not meet the condition that the cylinder keeps stress-less for a movement as rigid displacement of the shell in lateral direction. Next, the equation produces less reliable answers if wavelengths in circumferential direction become larger. This may not be a hindrance for shallow circular roof shells, but one has to be careful for chimneys.

Morley equation

Morley [4] improved the Donnell equation in 1959, replacing it by

$$\Delta\Delta(\Delta+1)^2 w + 4K^4 \frac{\partial^4 w}{\partial x^4} = \frac{1}{D}\Delta\Delta p \tag{4}$$

The equations (3) for u and v remain unchanged. Main physical backgrounds of the improvement are correction of the kinematic relation between the curvature $\kappa_{\theta\theta}$ in circumferential direction and the displacement w and adjusting the equilibrium equation in z-direction. Morley's approach considers a term w/a^2 in the curvature $\kappa_{\theta\theta}$ in addition to the second derivative with respect to w, which is the only part in Donnell's theory. Similarly Morley considers an additional term $m_{\theta\theta}/a^2$ in the equilibrium equation in z-direction, which Donnell does not. Even the Morley equation (4) is an approximation compared to the rigorous theory of Flügge, but it has sufficient completeness to produce far more accurate solutions than the Donnell equation. Moreover, the Morley equation now keeps the shell stress-free in case of a rigid body displacement and the equation remains valid for long wavelengths in circumferential direction.

2 Donnell application to chimneys under wind load

We consider a chimney which is considered to be fully clamped at the base and presume a wind load which is uniformly distributed over height and varies in circumferential direction, see Figure 2.

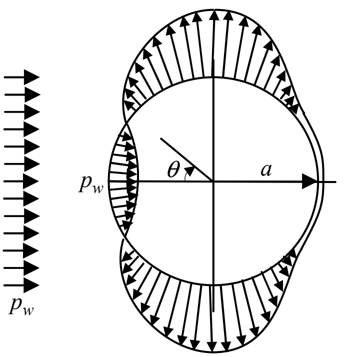

Figure 2. Typical distribution of the wind pressure

The wind load consists of pressure at the upwind face and heavy suction on the left and right sides of the chimney. The distribution of the wind pressure can be developed in a series

$$p(\theta) = -p_w(a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + a_4 \cos 4\theta + a_5 \cos 5\theta) \quad (5)$$

The value of the constant a_0 to a_5 depends on national codes and Reynolds number. The following values have been adopted, which are in agreement with [5] and [6].

$$a_0 = -0.823,\ a_1 = 0.448,\ a_2 = 1.115,\ a_3 = 0.400,\ a_4 = -0.113,\ a_5 = -0.027 \quad (6)$$

The axisymmetric term a_0 does not raise stresses in longitudinal directions , aside from insignificant stresses at the edges with very short influence length in comparison with the other terms. The $\cos\theta$ term with a_1 applies a varying pressure which has a maximum value at the windward meridian and a minimum value at the leeward meridian. This is the only term of the series which applies a horizontal resultant to the chimney due to the horizontal loading and therefore a bending moment at the base. Stresses produced by this loading conform to beam theory. For this loading term, plane sections remain plane and the circular section does not distort. The stress distribution is linear over the circular cross-section. All higher load terms cause vertical stresses which are self-equilibrating, but modify the longitudinal stresses produced by the $\cos\theta$ term. Particularly the $\cos 2\theta$ term causes “ovalizing” of the cross-section. Because this distortion cannot occur at the base, an equilibrating set of vertical stresses is evoked. As a result the maximum total tensile stress at the chimney base is larger than beam theory predicts. Figure 3 visualizes the joint effect

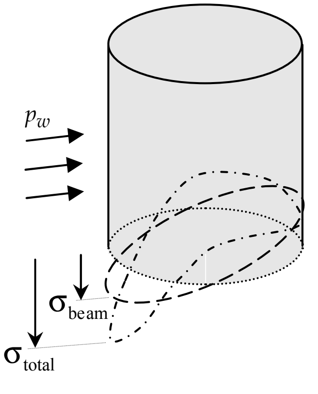

Figure 3. Distribution of the axial stress at the base

of the terms $\cos 2\theta$ to $\cos 5\theta$. It shows the distribution of the total stresses for all terms compared to the stress due to beam theory, viz. the term $\cos\theta$ only.

Van Koten [5] managed to derive an elegant expression for the ratio r between the maximum total tensile stress and the beam theory stress, in which two shell parameters l/a and t/a occur

$$r = 1 + \frac{k}{(\frac{l}{a})^2(\frac{t}{a})} \tag{7}$$

The dimensionless symbol k is a constant. Van Koten derived the expression (7) for r from the Donnell equation, but the same expression will be found applying the Morley equation. Just the value of k appears to become different. In working out, Van Koten restricted himself to the $\cos 2\theta$ term in the wind pressure series (constant a_2 = 1.115) and used zero Poisson's ratio. So, he obtained in the framework of Donnell k = 4.31. If he had considered all six terms of the series of Eq. (5) the value would only grow to 4.87, from which we notice that the $\cos 2\theta$ term in the wind pressure is indeed dominant as anticipated.

3 FE analysis

Two years after the publication of expression (7) for the ratio between total stress and beam theory stress Turner [6] has published a check through FE analyses. He computed a large number of shell configurations with extensively varying values of l/a and t/a as occur in practice. The value of l/a varied from 30 to 80 and the value of t/a from 0.004 to 0.012. He showed, applying all six terms in the wind pressure development (5) that the formula fits all his computations if he set the constant k to 6.05. Then the agreement between the formula (7) and the finite element results is within 0.5 percent for the large range of chimneys considered. So he confirmed the structure of the formula and the relevant parameters, however found a much larger value for the constant k.

4 Morley application to chimneys under wind load

Hoefakker [7] has reviewed the theories for cylindrical shells in his doctoral thesis. Among other activities, he repeated the computation of Van Koten for the determination of the constant on basis of the Donnell equation and achieved at the already mentioned k = 4.87. So, using the full series expansion of the wind load does not explain the much higher value

6.05 of Turner. Hoefakker furthermore applied the Morley equation to the chimney problem [7], confirming Van Koten's formula (7) for the ratio r and arriving at the general expression for k:

$$k = 4\sqrt{3(1-\nu^2)} \sum_{n=2}^{\infty} \frac{1}{n^2-1} \frac{a_n}{a_1} \tag{8}$$

Herein ν is Poisson's ratio and the mode number n depicts the respective wind load series term. Application of the series of constants of (6) results in the constant k

$$k = 6.39\sqrt{1-\nu^2} \tag{9}$$

Upon inquiry we learned that Turner has applied 0.3 for Poisson's ratio, which makes expression (9) resulting in k = 6.10, which is within one percent equal to the computational result 6.05 of Turner. The difference between k = 4.87 of Van Koten and Turner's value k = 6.05 is fully explained by the inaccuracy of the Donnell theory and use of different values of Poisson's ratio.

For the range of shell configurations considered by Turner we find r values ranging from r = 1.08 to r = 2.68. The lower value r = 1.08 holds for the combination of high parameter values t/a = 0.012 and l/a = 80. The higher value r = 2.68 holds for the combination of low parameter values t/a = 0.004 and l/a =30. This higher value 2.68 deviates substantially from the beam solution.

5 Discussion

The difference between the Donnell solution and Morley solution is easily understood from the different definitions of the curvature in circumferential direction. Using the coordinate $y = a\theta$ it holds

$$\begin{aligned} \text{Donnell}: \quad \kappa_{yy} &= -\frac{\partial^2 w}{\partial y^2} \\ \text{Morley}: \quad \kappa_{yy} &= -\frac{\partial^2 w}{\partial y^2} - \frac{w}{a^2} \end{aligned} \tag{10}$$

The minus signs are not essential but just occur because of chosen sign conventions. Let c be the wave length along the circumference in y-direction, see Figure 4. Then it holds

$$w(y) = \hat{w}\cos\frac{\pi y}{c} \tag{11}$$

and we obtain the following maximum curvatures

$$\text{Donnell}: \quad \hat{\kappa}_{yy} = \frac{\pi^2}{c^2}\hat{w}$$
$$\text{Morley}: \quad \hat{\kappa}_{yy} = (\frac{\pi^2}{c^2} + \frac{1}{a^2})\hat{w} \tag{12}$$

The difference between the two curvatures is negligible if

$$(\frac{c}{\pi a})^2 << 1 \tag{13}$$

Herein πa is the half circumference. For a wave length of one sixth of the half circumference the error in the Donnell curvature is less than 3 percent, for one fourth less than 6 percent, one third exactly 10 percent and one half already 20 percent, so rapidly increasing for longer wave lengths. If the wave length c approaches the half circumference the error is already 50 percent, so equal to the Donnell value itself.

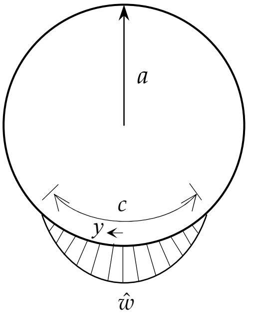

Figure 4. Definition of wave length c in circumferential direction

The highest ratio r for the shells considered by Turner is 2.68 on basis of the Morley equation and 2.35 by the Donnell equation, a difference of 14 percent. Not severe, however not negligible. The dominant term $\cos 2\theta$ in the wind pressure expansion has a wave length of one half of a half circumference and the next term $\cos 3\theta$. Here above we estimated 20 and 10 percent error respectively. The mentioned 14 percent between Donnell and Morley result is nicely in between these values.

Characteristic length

We can handle expression (7) for the ratio r in another way. Introducing the characteristic length $l_{ch} = \sqrt[4]{a\,t\,l^2}$ we can also write the expression as

$$r = 1 + k\left(\frac{a}{l_{ch}}\right)^4 \tag{14}$$

The fourth root is a characteristic length in circumferential direction. Instead of two parameters we now have one. In Turner's computational domain for shells in practice this characteristic length varies from 1.38 a to 2.96 a, which is 0.88 and 1.89 times the half circumference, respectively. For these values the difference between the Donnell equation and Morley equation is substantial. Of course, the calculated characteristic lengths will result in the same values of r ranging from 1.08 to 2.68.

6 Effect of ring stiffeners and axial elastic support.

With reference to Hoefakker [7] and author's *Structural Shell Analysis* [8], new design formulas, which describe the effect of stiffening rings and elastic supports, have been also developed based on the Morley equation. These have been presented such that the respective influence is represented by inclusion of an additional factor in the above formula (14) for the fixed base case without stiffening rings.

For distributed stiffening rings the stiffness ratio λ_r has been introduced, which is the ratio of the bending stiffness of the circular cylindrical shell only to the modified bending stiffness of the shell including the "smeared" contribution of the ring flexural stiffness. The formula for the case with stiffening rings reads

$$r = 1 + k\sqrt{\lambda_r}\left(\frac{a}{l_{ch}}\right)^4 \tag{15}$$

The design formula is applicable for sufficiently closely spaced stiffening rings, see [7]. A similar expression can be derived for an axial elastic support with stiffness k_x . Then the formula reads

$$r = 1 + k\sqrt{\lambda_{xn}}\left(\frac{a}{l_{ch}}\right)^4 \tag{16}$$

Herein the normalised ratio λ_{xn} is introduced, which depends on the respective factors a_n in the wind load development of expression (5), mode numbers n, the geometrical property a/t and the ratio of the axial elastic support stiffness k_x and the modulus of elasticity E. For the full description we refer to [7] and [8].

7 Summary and conclusion

The membrane tensile stress in axial direction at the base of a circular cylindrical chimney deviates from the stress due to classical beam theory. We considered two theories, from Donnell and Morley, respectively, to compute the total stress. For the ratio between the total stress and the beam theory stress the symbol r is used. The difference between the equations of Donnell and Morley seems small but can be of substantial meaning for stresses at the base of chimneys. For not very thin and long chimneys the difference is negligible. Then, beam theory will suffice. However, for really thin and rather low chimneys the difference is not negligible. For the specified wind pressure distribution (5) and the adopted combination of low parameter values t/a = 0.004 and l/a = 30, this difference is about 2.3 times the beam theory stress according to the Donnell equation and 2.7 according to the Morley equation.
Van Koten derived on basis of the Donnell equation an elegant formula for the ratio r between the total stress and the beam theory stress, which is confirmed by Hoefakker on basis of the Morley equation. The only difference is the value of the dimensionless constant k in the formula.
The results of the Morley equation have been happily confirmed based on finite element analyses of Turner. For a big range of shell configurations the difference appears less than one percent.
The explanation of the difference between the Donnell and Morley equation is easily done by a wave length consideration in circumferential direction. From this, the difference in the chimney results becomes plausible.

The formula for ratio r is conveniently extended for closely spaced ring stiffeners and an axial elastic support at the base.

References

[1] Love A.E.H., *The Mathematical Theory of Elasticity*, 4th edn, Cambridge University Press, Cambridge (1927).

[2] Flügge, W., *Statik und Dynamik der Schalen*, Berlin (1934).

[3] Donnell L.H., Stability of Thin-walled Tubes under Torsion. NACA Report No 479 (1933).

[4] Morley L.S.D., An Improvement on Donnell's approximation for Thin-walled circular cylinders, *Quarterly of Mechanics and Applied Mathematics*, 12 (1959).

[5] Van Koten, H., The Stress Distribution in Chimneys due to Wind Pressure, CICIND Copenhagen Meeting (1994).

[6] Turner, J.G., Wind Load Stresses in Steel, CICIND, Orlando Meeting (1996).

[7] Hoefakker J.H., *Theory Review for Cylindrical Shells and Parametric Study of Chimneys and Tanks*, Eburon Academic Publishers, Delft (2010).

[8] Blaauwendraad J., Hoefakker J.H. (2014), *Structural Shell Analysis*, Springer.