Moments due to concentrated loads on thin shell structures

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Formulas are presented for the moments in thin shell structures due to concentrated loads perpendicular to the surface. The moments are a function of the shell curvature, the shell thickness and the area over which the force is distributed. The shell edges have no influence provided these are at a sufficiently large distance from the concentrated load. The formulas have been derived from 55 linear elastic finite element analyses of shells with positive and negative Gaussian curvatures.

Keywords: Thin shell structure, point load, bending moment, elasticity theory, finite element method, design formula

1 Introduction

In well-designed shell structures most of the load is carried by membrane forces and the moments are small. Nevertheless, moments occur at edges, at discontinuities and at concentrated loads. These moments occur locally and are negligible at some distance from their peak value (fig. 1). Clearly, these moments and the whole force flow in a shell structure can be computed by finite element analyses. However, it is useful to have simple formulas for often occurring situations. For concentrated loads such formulas can be derived because most of the deformation of a thin shell is restricted to the local area around the concentrated load and therefore depends on only a few parameters.

An obvious way to derive these formulas is from the mathematical equations that describe shell behaviour. The general equations have been derived by Sanders and Koiter in 1959 [1, 2]. Unfortunately, these equations are large and can only be solved by introducing smart approximations which is very difficult. Nonetheless, a simplified set of equations has been solved for concentrated loads on spherical caps by Reissner in 1946 [3]. He showed that the deformation of these shells is described by Kelvin functions. The deflection w and peak values of moments m_{xx} , m_{yy} and normal forces n_{xx} , n_{yy} can be derived from his work.

$$
\begin{aligned}
w &= \frac{\sqrt{3}}{4}\frac{Pa}{Et^2}\sqrt{1-\nu^2} \\
m_{xx} = m_{yy} &= \frac{1}{8\sqrt[4]{12}}\frac{P\sqrt{at}}{d}\frac{1+\nu}{\sqrt[4]{1-\nu^2}} \\
n_{xx} = n_{yy} &= -\frac{\sqrt{3}}{8}\frac{P}{t}\sqrt{1-\nu^2}
\end{aligned}
\tag{1}
$$

The notation is explained in the next chapter. The correctness of these formulas has been confirmed by finite element analyses, except for the moment formula for which there must be some mistake in the derivation.
Between 1969 and 1978 several papers were published on other shells shapes with concentrated loads such as cylinders [4], elpars [5], hypars [6] and shells of arbitrary shape [7, 8]. The results of these analytical studies were presented as infinite series and numerical examples. That work did not lead to useful design formulas.

Instead of an analytical approach the work presented in this paper followed a computational approach. The finite element software Ansys was used with shell elements [9] that are derived from solid elements without any use of the Sanders-Koiter equations. A script was written to generate finite element models with different parameters such as curvatures, thickness, length and width. Computed and recorded were the moments at the position of the concentrated load. Curves were fitted through these moments providing formulas and parameter ranges for formula validity.

It can be called a pleasant surprise that the derived formulas are simple despite the complexity of shell mathematics. This suggests that shell behaviour is not as difficult as often thought.

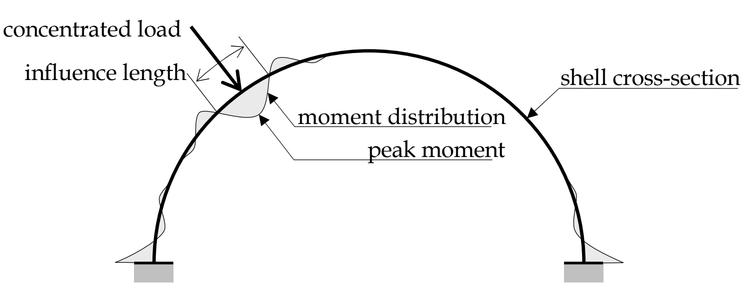

Figure 1. Moment distribution in a shell cross-section due to a concentrated load (Proportions are exaggerated to clearly show the phenomenon.)

2 Finite element analyses

Consider a thin shell structure with varying shape as shown in figure 2. The shell is loaded by a force P perpendicular to the surface. The force is evenly distributed over small circular area with diameter d.[1] The shell thickness t is constant over the surface. A Cartesian coordinate system is placed in the middle surface where the load is applied. The z axis is in the direction of the load. In the x direction the shell has a curvature k_{xx} in the origin. In the y direction the shell has a curvature k_{yy} in the origin. The curvatures can be negative or positive. There is no twist curvature k_{xy} in the origin, therefore, the coordinate system is aligned with the principal curvatures.[2] The shell middle surface is described by a paraboloid.

$$z = \frac{1}{2} k_{xx} x^2 + \frac{1}{2} k_{yy} y^2 \tag{2}$$

This paraboloid can be interpreted as a second-order Taylor expansion of the real shell surface around the concentrated load. The shell edge is supported by rollers that carry normal forces and in-plane shear forces only (fig. 3). The material behaviour is linear elastic.

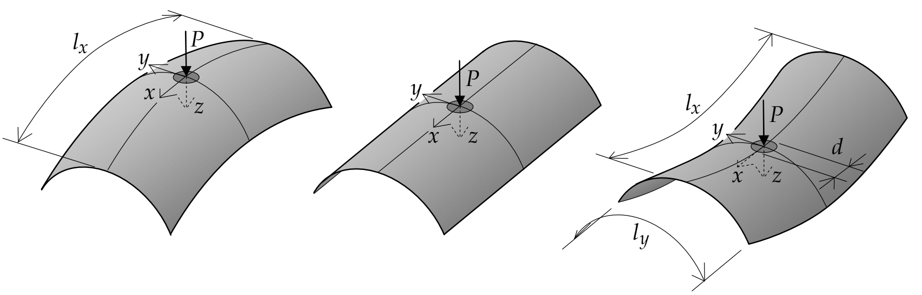

Figure 2. Shell shapes with concentrated loads and coordinate systems

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[1] If the load is not distributed the out-of-plane shear force is infinite in the origin. This leads to infinite moments and an infinite deflection. The latter is not due to bending deformation but due to out-of-plane shear deformation.

[2] A zero value of k_{xy} is not a restriction because the coordinate system in a shell point can always be chosen in the principal curvature directions without loss of generality.

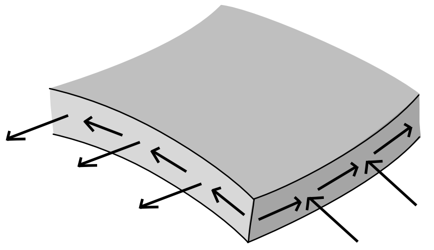

Figure 3. Corner of a shell with the edge support reactions

The following notation is used in this paper.

a …………………	radius of curvature ……………………………………………	[m]
d ……………….....	load distribution diameter ………………………………...	[m]
E …………………	Young's modulus ……………………………………………..	[N/m²]
f …………………	function ……………………………………………………...	[-]
k_{xx} , k_{yy} …………	curvatures in the origin in the x and y directions ………	[1/m]
l_x , l_y ………………	lengths of the shell measured over the middle surface ..	[m]
m_{xx} , m_{yy} , m_{xy} …	moments in the origin ……………………………………..	[Nm/m]
n_{xx} , n_{yy} , n_{xy}	membrane forces in the origin ……………………………	[N/m]
P ………………..	concentrated load ……………………………………………..	[N]
t …………………	shell thickness …………………………………………………	[m]
w ………………..	deflection ………………………………………………………	[m]
x, y, z …………..	Cartesian coordinates ……………………………………...	[m]
ν …………………	Poisson's ratio …………………………………………….......	[-]

A hypar shell shape has been entered in the finite element program Ansys [9]. The following input has been used, $E = 10^5$ N/mm², ν = 0.0, $k_{xx} = -1/(3000$ mm$)$, $k_{yy} = 1/(3000$ mm$)$, $t = 1$ mm, $d = 5$ mm, $P = 50$ N. The shell length and width are $l_x = l_y = 1315$ mm. These are ten times the estimated influence length of $2.4\sqrt{at}$. To reduce computation time only 1/4 of the shell has been modelled (fig. 4). The applied shell finite elements have 8 nodes and quadratic shape functions (SHELL281) [9]. The element size is 0.25 mm in the origin and 23 mm at the boundary. It has been verified that half the element size gives the same results. It has been verified that a double shell length and width give the same results. The output is shown in figures 4, 5 and 6.

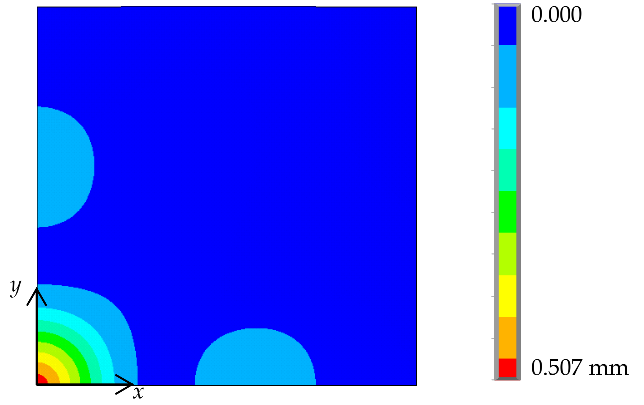

Figure 4. Deformation of the hypar shell (displacement length)

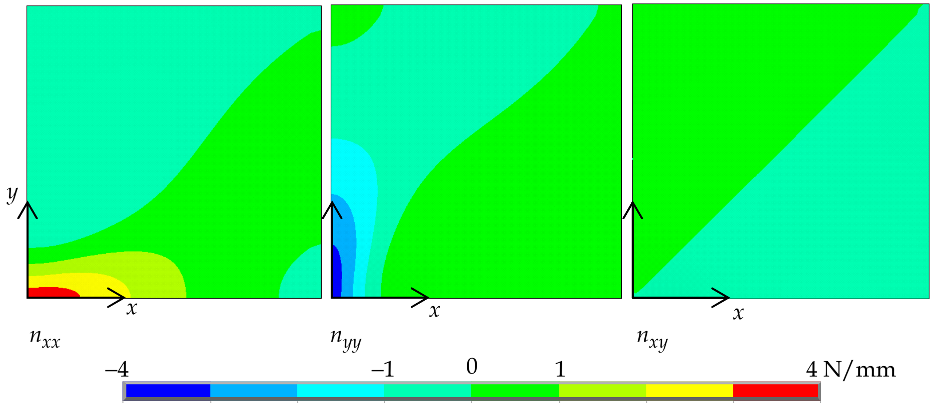

Figure 5. Membrane forces in the hypar shell

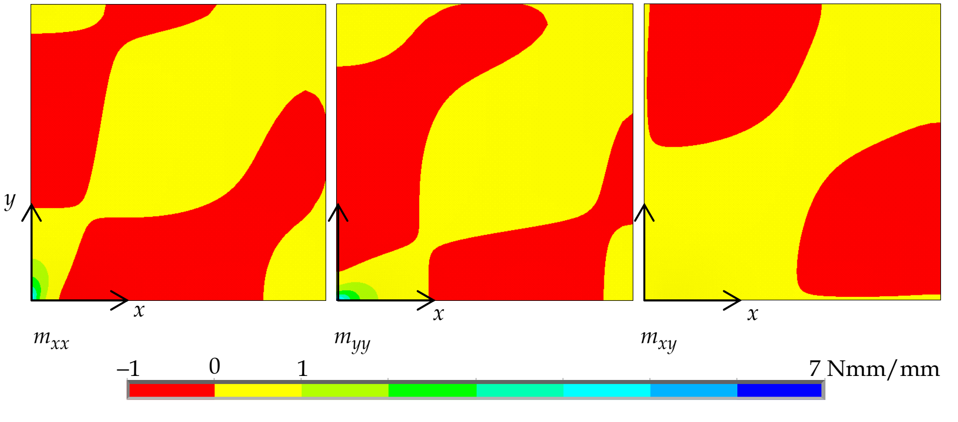

Figure 6. Moments in the hypar shell (colour figures are available at www.heronjournal.nl)

The results also show that torsion moments m_{xy} do not occur in the origin. Apparently, in the origin, the principal moments are perfectly aligned with the principal curvatures. The input has been varied 54 times. The moments under the point load are presented in chapter 4.

3 Data analyses

In this section the approach for obtaining the moment tables is explained.

Since a linear elastic model is used the moments m_{xx} , m_{yy} are proportional to the load P. The moments are independent of Young's modulus E because there are no volume changes such as temperature load and no imposed displacement unequal to zero. In previous studies it was found that Poisson's ratio ν has an influence on the results, however, usually this influence is small. Therefore, initially it was assumed that ν = 0. Thus, the remaining variables are thickness t, load diameter d, curvature k_{xx} and curvature k_{yy} . There is a function f such that

$$m_{xx} = P\, f_1(t, d, k_{xx}, k_{yy}) \,. \tag{3}$$

This equation can be rewritten as

$$\frac{m_{xx}}{P} = f_2(t, \frac{d}{t}, k_{xx}t, k_{yy}t) \,. \tag{4}$$

The units are $\frac{m_{xx}}{P}$ [–], t [m], $\frac{d}{t}$ [-], $k_{xx}t$ [–], $k_{yy}t$ [–]. Consequently, this equation cannot be correct because a dimensionless result cannot be obtained when one of the variables has a dimension. A correct equation can be

$$\frac{m_{xx}}{P} = f_3(\frac{d}{t}, k_{xx}t, k_{yy}t) \,. \tag{5}$$

The geometry of the problem provides more help in deriving function f_3 . When the values of k_{xx} and k_{yy} are exchanged the shell looks like rotated over 90° around the z axis. The moments are exchanged too, therefore,

$$\frac{m_{yy}}{P} = f_3(\frac{d}{t}, k_{yy}t, k_{xx}t) \tag{6}$$

When the values of k_{xx} and k_{yy} both change sign the shell looks like it has rotated 180 ^{o} around the x axis or y axis. It looks like both the load and the z axis have changed in direction. Consequently, the moments do not change

$$\frac{m_{xx}}{P} = f_3(\frac{d}{t}, -k_{xx}t, -k_{yy}t) \tag{7}$$

In addition, when $|k_{xx}t|$ or $|k_{yy}t| > 1/30$ the structure is commonly regarded a thick shell, which is outside the scope of this paper. When $|k_{xx}t|$ and $|k_{yy}t| < 1/3000$ the structure is a membrane or a flat plate, which is also outside the scope of this paper. Thus, just the grey part in figure 7 needs considering.

4 Finite element results

Finite element analyses have been performed of shells of various shapes. To this end an Ansys script has been developed. This script is available at heronjournal.nl/61-3/script.txt. The tables 1 and 2 show the response under the concentrated load as a function of k_{xx} and k_{yy} for the case $d = 5t$ and $\nu = 0$. Diameter d of the loaded area has been varied in the range $t < d < 5\ t$. Poisson's ratio ν has been varied in the range $0 < \nu < 0.5$. These have an influence on the tables, which is included in the formulas of chapter 5.

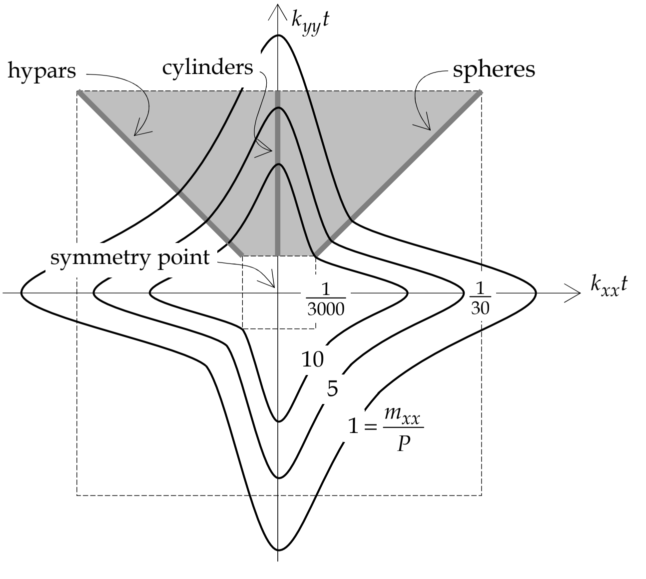

Figure 7. Possible contour lines of $\frac{m_{xx}}{P}$

Table 1. Moment m_{xx} as a function of curvature, ν = 0, d = 5t

$\frac{m_{xx}}{P}$		$k_{yy}\,t$				
		$\frac{1}{3000}$	$\frac{1}{1000}$	$\frac{1}{300}$	$\frac{1}{100}$	$\frac{1}{30}$
	-1.0	0.265	0.221	0.173	0.130	0.0838
	-0.8	0.265	0.222	0.174	0.131	0.0849
	-0.6	0.266	0.222	0.174	0.131	0.0855
	-0.4	0.265	0.222	0.174	0.131	0.0855
	-0.2	0.264	0.220	0.173	0.130	0.0846
$\frac{k_{xx}}{k_{yy}}$	0.0	0.260	0.217	0.169	0.126	0.0816
	0.2	0.256	0.213	0.165	0.122	0.0773
	0.4	0.253	0.209	0.161	0.119	0.0738
	0.6	0.250	0.206	0.159	0.116	0.0711
	0.8	0.248	0.204	0.156	0.114	0.0688
	1.0	0.245	0.202	0.154	0.112	0.0668

Table 2. Moment m_{yy} as a function of curvature, ν = 0, d = 5t

$\frac{m_{yy}}{P}$		$k_{yy}\,t$				
		$\frac{1}{3000}$	$\frac{1}{1000}$	$\frac{1}{300}$	$\frac{1}{100}$	$\frac{1}{30}$
	-1.0	0.265	0.221	0.173	0.130	0.0838
	-0.8	0.272	0.229	0.181	0.137	0.0906
	-0.6	0.281	0.237	0.189	0.146	0.0984
	-0.4	0.290	0.247	0.199	0.155	0.108
	-0.2	0.301	0.257	0.209	0.166	0.118
$\frac{k_{xx}}{k_{yy}}$	0.0	0.300	0.256	0.208	0.165	0.118
	0.2	0.284	0.241	0.193	0.150	0.102
	0.4	0.270	0.227	0.179	0.136	0.0891
	0.6	0.260	0.216	0.169	0.126	0.0797
	0.8	0.252	0.208	0.161	0.118	0.0725
	1.0	0.245	0.202	0.154	0.112	0.0668

5 Formulas

The moments under a concentrated load perpendicular to a shell surface are approximately

$$
\begin{aligned}
m_{xx} &= 0.0388(1+\nu)P\ln\frac{t}{\tilde{k}_{yy}d^2}\ ,\\
m_{yy} &= 0.0388(1+\nu)P\ln\frac{t}{\tilde{k}_{xx}d^2}\ ,\\
m_{xy} &= 0\,,
\end{aligned}
\tag{8}
$$

where,

$$
\begin{aligned}
\tilde{k}_{yy} &= \left|0.00725\,k_{xx}+0.119k_{yy}\right|+\left|\,0.0529k_{xx}+0.0298\,k_{yy}\right|,\\
\tilde{k}_{xx} &= \left|0.00725\,k_{yy}+0.119k_{xx}\right|+\left|\,0.0529k_{yy}+0.0298\,k_{xx}\right|.
\end{aligned}
$$

The deviation from the computational results is less than 10%. The deviation mainly occurs for the cylinder shapes for which the formula underestimates the moments. For hypars shapes the formula overestimates the moments. The curvature domain for which the formulas are valid is

$$
\begin{aligned}
&\left|k_{xx}t\right| > \frac{1}{3000} \text{ or } \left|k_{yy}t\right| > \frac{1}{3000}\\
&\text{and}\\
&\left|k_{xx}t\right| < \frac{1}{30} \text{ and } \left|k_{yy}t\right| < \frac{1}{30}\,.
\end{aligned}
\tag{9}
$$

The load diameter domain for which the formulas are valid is

$$
0 \le \frac{d}{t} < 5\,. \tag{10}
$$

The x and y axis are in the principal curvature directions, therefore, $k_{xy} = 0$. The formulas are not valid if boundaries are close to the concentrated load, such as a shell edge, a change in thickness or a significant change in curvature. The formulas are not valid for large deflections. A large deflection can already be $w > \frac{1}{2}t$. The formulas are not valid in case of in-extensional deformation.

6 Two application examples

A metal shell structure is loaded by a concentrated load of 750 N perpendicular to its surface. The shell has and ellipsoidal shape. The radii of curvature at the concentrated load are 500 mm and 2000 mm. The radii are in the principal curvature directions. The shell thickness is 2 mm. The concentrated load is distributed over a circular area with a diameter of 7 mm. The closest shell edge is at 300 mm from the location of the concentrated force. The shell edges are firmly supported such that in-extensional deformation does not occur. Young's modulus E is $2.1 \cdot 10^5$ N/mm². Poison's ratio ν is 0.35. The yield strength is 400 N/mm². The above values are design values, in other words, partial safety factors are included. Stresses due to self-weight and other loads can be neglected.

The moments under the concentrated load are calculated by equations (8).

$$\begin{aligned} m_{xx} &= 193 \text{ Nmm/mm} \\ m_{yy} &= 213 \text{ Nmm/mm} \\ m_{xy} &= 0 \end{aligned} \tag{11}$$

The membrane forces under the concentrated load are calculated by Ansys. (For computing the membrane forces the finite elements do not need to be small. However, for calculating the peak moments accurately the element size would have to be very small; smaller than 1/10 of the load diameter d.)

$$\begin{aligned} n_{xx} &= -141 \text{ N/mm} \\ n_{yy} &= -153 \text{ N/mm} \\ n_{xy} &= 0 \end{aligned} \tag{12}$$

The stress in the top shell surface ($z = - t/2$) is

$$\begin{aligned} \sigma_{xx} &= -6\frac{m_{xx}}{t^2} + \frac{n_{xx}}{t} = -290 - 71 = -360 \text{ N/mm}^2, \\ \sigma_{yy} &= -6\frac{m_{yy}}{t^2} + \frac{n_{yy}}{t} = -319 - 77 = -396 \text{ N/mm}^2, \\ \sigma_{xy} &= -6\frac{m_{xy}}{t^2} + \frac{n_{xy}}{t} = 0 . \end{aligned} \tag{13}$$

The stresses in the bottom shell surface are smaller. The Von-Mises stress is

$$\sigma_{VM} = \sqrt{\tfrac{1}{2}\left((\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2\right)+3\left(\sigma_{xy}^2+\sigma_{xz}^2+\sigma_{yz}^2\right)} \\ = 379 \text{ N/mm}^2 \tag{14}$$

The Von-Mises stress is smaller than the yield stress, therefore, the material will not yield due to the concentrated force.

A reinforced concrete shell structure is loaded by a concentrated load of 18 kN perpendicular to its surface. The shell has a hypar shape and at the concentrated load the radii of curvature are 100 m and 200 m in the principal curvature directions (k_{xx} = 1/(100 m), k_{yy} = –1/(200 m). The shell thickness is 100 mm. The concentrated load is distributed over a circular area with a diameter of 100 mm. The closest shell edge is at 15 m from the location of the concentrated load. The shell edges are firmly supported such that in-extensional deformation does not occur. Poisson's ratio of the cracked reinforced concrete is set to 0. The moments under the concentrated load are calculated by equations (8).

$$m_{xx} = 6538 \text{ Nmm/mm}$$
$$m_{yy} = 6221\text{Nmm/mm}$$
$$m_{xy} = 0 \tag{15}$$

The membrane forces under the concentrated load are calculated by Ansys.

$$n_{xx} = 16 \text{ N/mm}$$
$$n_{yy} = -33 \text{ N/mm}$$
$$n_{xy} = 0 \tag{16}$$

Proposed are reinforcing bars with a diameter ϕ = 12 mm and a spacing s = 200 mm (fig. 8). The bars are applied in the middle of the shell thickness in the principal curvature directions. The yield stress of the reinforcing bars f_y = 400 N/mm². The compressive strength of the concrete is 30 N/mm². Stresses due to self-weight and other loads are neglected.

The cross-section capacity can be checked as follows. [3]

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[3] The calculation uses an estimate of the concrete stress *fc* and an estimate of the concrete compression zone *e*. This is based on the lower bound theorem of plasticity theory; Any equilibrium system – with stresses that are not too large – gives a safe approximation of the capacity of a ductile structure [10].

bar yield force $N = \frac{1}{4}\pi\phi^2 f_y$ = 45200 N,

$n_{xp} = \frac{N}{s} - \frac{1}{2} f_c\, e$ = 16 N/mm, if f_c = 14 N/mm² and e = 30 mm

$$m_{xp} = -\frac{N}{s}\frac{\phi}{2} + \frac{1}{2} f_c\, e(\tfrac{1}{2}t - \tfrac{1}{3}e) = 7040 \text{ Nmm/mm} > 6538 \text{ Nmm/mm} \tag{17}$$

$n_{yp} = \frac{N}{s} - \frac{1}{2} f_c\, e$ = –33 N/mm, if f_c = 20.7 N/mm² and e = 25 mm

$m_{yp} = \frac{N}{s}\frac{\phi}{2} + \frac{1}{2} f_c\, e(\frac{1}{2}t - \frac{1}{3}e)$ = 12100 Nmm/mm > 6221 Nmm/mm

Consequently, the proposed reinforcement is sufficient. Punching shear and crack widths are not checked in this example.

If the reinforcing bars would not be in the principal curvature directions then the shell moments would need to be transferred to the reinforcement directions s, t by Mohr's circle and the required capacities would need to be calculated by the Wood-Armer moments $m_{ss} + |m_{st}|$ and $m_{tt} + |m_{st}|$ [11]. If the reinforcement design is computer supported then a more advanced plate model can be applied such as the three layer sandwich model of the Eurocode [12].

7 Conclusions

A concentrated force perpendicular to a shell surface causes moments which are largest directly under the force. The principal directions of these moments are aligned with the principal curvature directions. The moments in the two principal directions are roughly the same and the largest of the two curvatures has a dominant influence on the magnitude of these moments. The peak moments are approximately the same for spherical, cylindrical and hypar shells of the same curvature. Spherical shells have the smallest moment; hypar shells have a somewhat larger moment and cylindrical shells have substantially larger

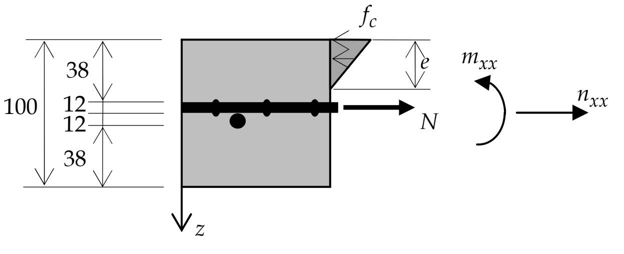

Figure 8. Reinforcement in the middle of the shell thickness

moment (See table 1 and 2).

The moments can be predicted by simple formulas, which are valid for a large range of shell curvatures and shell thicknesses. A larger shell thickness results in a larger peak bending moment (See eq. 8). The extra thickness reduces the stresses but not as much as for moments in plates. The stresses due to the moments are considerably larger than the stresses due to the membrane forces.

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