Conical shell edge disturbance
An engineer's derivation

J. Blaauwendraad

Emeritus-Professor in Structural Mechanics, Faculty of Civil Engineering and Geosciences, Delft University of Technology, the Netherlands

J.H. Hoefakker

Former lecturer in Structural Mechanics, Faculty of Civil Engineering and Geosciences, Delft University of Technology, the Netherlands

Because a rigorous bending theory for thin shells of revolution is complicated, attempts have been made for reliable approximations of the edge disturbance problem under axisymmetric loading. A well-known one was published by Geckeler [1, 2], who obtained his approximation by mathematical considerations. He started from kinematic, constitutive and equilibrium equations for the rotationally symmetric thin shell without approximations. Herein he introduced mathematical simplifications. Each time when derivatives of a function of different orders appeared, he just kept the highest order derivative and neglected all lower ones. This is permitted if the function varies rapidly, as is the case for edge disturbances. Here we will present Geckeler's result in an alternate way, which illustrates the physical background of his mathematical approximation. Said in another way, we offer a derivation in the language of structural engineers.

Keywords: Rotationally symmetric shell, edge disturbance zone, engineer's approach

1 Introduction

In Figure 1 four shells of revolution are shown: a circular cylindrical shell, a conical one a sphere half and a spherical cap. The sphere half has a cylinder as envelope and the spherical cap a cone. The envelope is the tangent plane at the base circle. We expect the reader to be familiar with the solution of edge disturbances in the circular cylindrical shell. The disturbance really occurs in a small edge zone of about three times $\sqrt{at}$ in which a is the radius and t the thickness of the cylindrical shell. So, the length of the edge zone is always small compared to the length or height of thin shells. As is well known, this means that we can use the solution of the cylinder in Figure 1 also for the sphere half; after all, at the edge, the cylinder is the tangent plane of the sphere half, and the difference between the small edge zone of sphere and cylinder is negligible. For the cone we will have another solution than for the cylinder, but again we can expect a disturbance that is limited to a similar edge zone. Having solved the cone problem, one can also use this solution for the spherical cap. Therefore, we will focus in this article on the cone problem. We will limit ourselves to the derivation of the edge disturbance theory. For applications to particular cases of distributed shell loads is referred to standard text books like author's *Structural Shell Analysis* [3]. Hereafter we will discuss the accuracy of the approximation for cones and spherical caps.

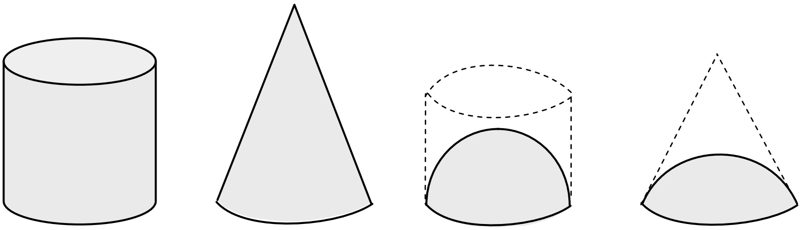

Figure 1. Four different shells of revolution

2 Recall for circular cylinder

Though the solution to the edge disturbance problem of circular cylindrical shells is well-known, we will recall it because we will use the same procedure of derivation for the cone problem. We first present the solution for the shell as depicted in Figure 2. The length of the shell in x-direction is supposed to be large compared with the length of the edge disturbance zone. We choose the origin of the x-axis at the edge of the shell. In the same figure we show the sign convention for the bending moments m_{xx} and $m_{\theta\theta}$, transverse shear force v_x and circumferential membrane force for $n_{\theta\theta}$. We focus on circular cylindrical shells with an edge for $x = 0$ under axisymmetric loading. We put p_x to zero, so consider only a normal load p_z . The final result is a superposition of the membrane solution and bending solution. The membrane solution accounts for the distributed load p_z on the shell, and the bending solution accounts for edge loads at the base of the shell in order to satisfy the boundary constraints. Because of symmetry considerations, no circumferential displacement does occur and no shear membrane strain, shear membrane force and twisting moment need be considered. The normal membrane force n_{xx} can be neglected as it is related to the membrane state. The change of curvature κ_{xx} will not be zero, which has to vary with respect to the ordinate x in order to overcome any discrepancies between the boundary conditions at the base and the membrane displacements of the shell. Accordingly, the circumferential normal membrane force $n_{\theta\theta}$ and bending moment $m_{\theta\theta}$ are also activated; they are constant in circumferential direction because of the axisymmetric conditions. We apply Donnell's theory for thin shallow shells [4], which implies that the circumferential bending moment $m_{\theta\theta}$ can only develop due to the effect of Poisson's ratio. The circumferential curvature is zero, therefore $m_{\theta\theta} = \nu m_{xx}$, where ν is Poisson's ratio.

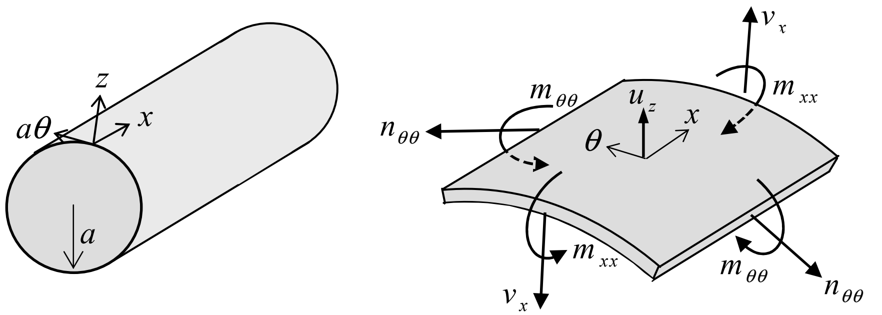

Figure 2. Circular cylindrical shell; Stress resultants; Sign convention

2.1 ***Derivation of differential equation***

Accounting for all these expectations, we model a circular cylindrical shell in axisymmetric bending as a barrel consisting of vertical staves and horizontal ring belts, see Figure 3. The staves are straight and have unit width, and classical Euler-Bernoulli beam theory applies, except that the modulus of elasticity E is replaced by $E/(1 - \nu^2)$ in order to account for lateral contraction. The normal displacement u_z , the circumferential membrane force $n_{\theta\theta}$, bending moments m_{xx} and $m_{\theta\theta}$, and transverse shear force v_x are functions of the coordinate x only.

If the staves tend to displace outward, the rings will be strained. A uniformly distributed interaction force q_z acts outward on each ring and inward on the staves. The differential equation for the stave is

$$D\frac{d^4u_z}{dx^4} = p_z \tag{1}$$

where D is the *flexural rigidity*

$$D = \frac{Et^3}{12(1-\upsilon^2)} \tag{2}$$

We now consider the rings of unit width. The force q_z is proportional to the radial displacement u_z . Due to the displacement the ring radius will increase. Load q_z acts on the rings in positive z-direction and on the staves in negative direction. Beam equation (1) changes into

$$D\frac{d^4u_z}{dx^4} = p_z - q_z \tag{3}$$

For the relation between q_z and the radial displacement u_z , it holds

$$q_z = \frac{Et}{a^2}u_z \tag{4}$$

We substitute Eq. (4) in Eq. (3) and obtain the differential equation that governs the edge disturbance.

$$D\frac{d^4u_z}{dx^4} + \frac{Et}{a^2}u_z = p_z \tag{5}$$

As known, the differential equation is fully similar to the equation for a beam on an elastic foundation. The term Et/a^2 is the spring stiffness.

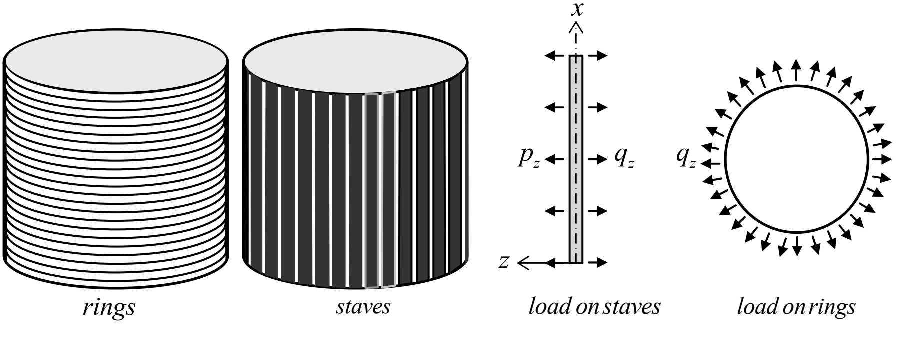

Figure 3. Rings and staves in a circular cylindrical shell

From here we put $p_z = 0$ and introduce the positive parameter β defined by:

$$\beta^4 = \frac{E\,t}{4\,D\,a^2} = \frac{3(1-\nu^2)}{(a\,t)^2}\,. \tag{6}$$

The homogeneous equation becomes

$$\frac{d^4 u_z}{dx^4} + 4\beta^4 u_z = 0 \tag{7}$$

The solution for an edge disturbance is:

$$u_z(x) = C\,e^{-\beta x}\sin(\beta x + \psi) \tag{8}$$

in which C and ψ are constants of dimensions displacement and angle respectively. The solution is an oscillating function of x that decreases exponentially with increasing x. The function has pleasant characteristics for derivation. Each following derivative means subtraction of π/4 in the sine argument and multiplication of the function by √2. Accounting for the sign conventions in Figure 2 and elementary parts in Figure 4 we arrive at the following bending moments and transverse shear force in the staves and circumferential membrane forces in the rings:

$$\begin{aligned} m_{xx} &= -D(2\beta^2\,C\,e^{-\beta x}\sin(\beta x + \psi - \tfrac{\pi}{2})) \\ m_{\theta\theta} &= \nu\,m_{xx} \\ v_x &= -D(-2\sqrt{2}\,\beta^3\,Ce^{-\beta x}\sin(\beta x + \psi - \tfrac{3\pi}{4})) \\ n_{\theta\theta} &= \frac{E\,t}{a}Ce^{-\beta x}\sin(\beta x + \psi) \end{aligned} \tag{9}$$

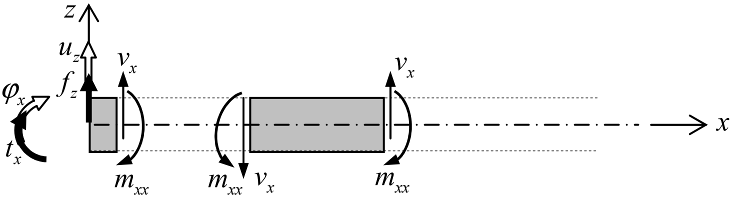

Figure 4. Elementary boundary part at x = 0 (left) and elementary shell wall part (right)

The displacement and rotation at the shell end x = 0 are

$$u_z = C \sin \psi$$
$$\varphi_x = \sqrt{2}\,\beta C \sin(\psi - \tfrac{\pi}{4}) \tag{10}$$

where φ_x is a positive rotation with respect to the $a\theta$-axis in Figure 2.

2.2 *Edge force and torque*

Consider the two semi-infinitely long shells in Figure 5 which are loaded at the free end x = 0 per unit length by an edge force f_z and edge torque t_x respectively. When loaded by f_z the rotation φ can develop freely and when loaded by the torque t_x the displacement u_z can occur freely. The constants C_f and ψ_f for load f_z and C_t and ψ_t for load t_x become respectively

$$\text{Force } f_z: \quad \psi_f = \tfrac{\pi}{2}; \quad C_f = \frac{f_z}{2D}$$
$$\text{Torque } t_x: \quad \psi_t = \tfrac{3\pi}{4}; \quad C_t = \frac{t_x}{\sqrt{2}D\beta^4} \tag{11}$$

With aid of these constants we derive from Eq. (10) the *flexibility* matrix equation for the general case that the edge force and edge torque occur both.

$$\frac{1}{D}\begin{bmatrix} \frac{1}{2\beta^3} & \frac{1}{2\beta^2} \\ \frac{1}{2\beta^2} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} f_z \\ t_x \end{bmatrix} = \begin{bmatrix} u_z \\ \varphi_x \end{bmatrix} \tag{12}$$

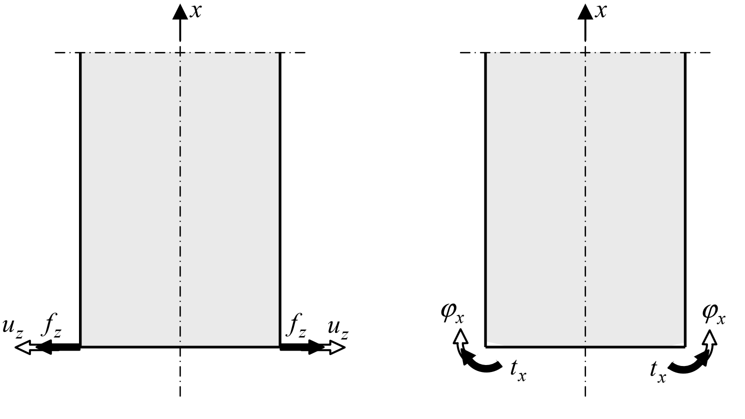

Figure 5. Edge force f_z and torque t_x, displacement u_z and rotation φ_x

Eq. (12) facilitates satisfying the boundary equations at the edge for each loading case in which both a membrane state and edge disturbance occur. After the size of the edge loads f_z and t_x have been determined, the moments and forces in the edge zone can be derived from Eq. (9), accounting for Eq. (11)

3 Extension to cones

Consider the cone of Figure 6 with base circle of radius a. We will use the concept of staves and rings again. There are a number of differences with the application to circular cylinders:

1. The staves are not prismatic but tapered.
2. The radius of the rings is not constant.
3. In cylinders the membrane force n_{xx} in axial direction is zero, but not in cones.
4. For cylinders the displacement u_z is in ring radius direction, but not for cones.

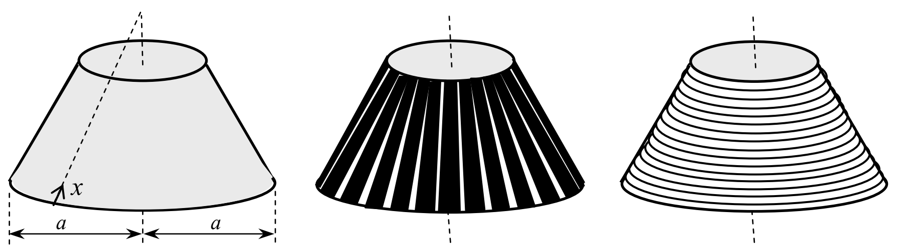

Figure 6. Cone with chosen x-axis, staves and rings

The simplifying assumptions will regard the items 1 on taper and 2 on ring radius change. The disturbance remains limited to a narrow zone near the edge, therefore the effect of tapering will be very small. We neglect the taper and consider the staves to be prismatic. For the same reason we neglect the change of ring radius. All rings in the disturbed edge zone are supposed to have the radius a of the base circle.

Item 3, the existence of nonzero membrane forces in x-direction, implies strains ε_{xx} and therefore nonzero displacements in x-direction. However, these displacements are of the membrane type and an order of magnitude smaller than the displacements u_z due to bending in the disturbance zone. Therefore, we can neglect displacements in x-direction. To cope with item 4, we introduce in Figure 7 the angle ϕ to define the inclination of the cone wall. We choose an x-ordinate along the straight meridian, starting at the cone base. The z-axis is outward normal to the cone wall and so is the displacement u_z . Again we introduce the interaction force between staves and rings. Here above, we called this force q_z because it was directed in the normal direction z; now it does not, because it is in the direction of the ring radius r in the horizontal plan. Therefore we name the interaction force now q_r .

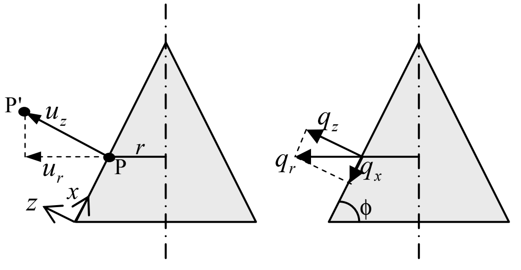

Figure 7. Definition of ring quantities u_r and q_r in a cone

3.1 ***Derivation of differential equation***

Consider in Figure 7a point P at the shell wall. The point is supposed to be in the edge disturbance zone. The shell part above the edge zone does not replace. The displacement of this point consists of a normal displacement u_z only. Then, the new position is P' and the increase u_r of the radius r due to the normal displacement is

$$u_r = u_z \sin \phi \tag{13}$$

The force q_r to produce this displacement is, equivalent to Eq. (4),

$$q_r = \frac{Et}{r^2} u_r \tag{14}$$

This force acts on the stave under an angle ϕ. We must decompose it in two components, a component q_z normal to the stave in negative z-direction and a component in the x-direction. The latter is related to the membrane state causing deformations to be neglected, and the first provides the elastic support of the stave. It holds that

$$q_z = q_r \sin \phi \tag{15}$$

We substitute Eq. (14) in Eq. (15), accounting for Eq. (13) and noticing $r_x \approx a$,

$$q_z = k_z u_z; \quad k_z = \frac{Et}{a^2} \sin^2 \phi \tag{16}$$

The behaviour of the stave remains unchanged compared to the circular cylinder. We conclude that the parallelism with the 'beam on elastic foundation' remains valid; the only difference with the circular cylinder is the calculation of the spring constant k_z . Now the additional factor $\sin^2 \phi$ comes in, which is unity for $\phi = \pi/2$, the value for a circular cylinder. The 'elastic foundation' of cones is less stiff than of circular cylinders. In fact, the message of Eq. (16) is that we must replace radius a by the principal radius r_2 as shown in Figure 8. From the left part of the figure we obtain $a = r_2 \sin \phi$, which changes Eq. (16) into

$$q_z = k_z u_z; \quad k_z = \frac{Et}{r_2^2} \tag{17}$$

If we replace β of Eq. (6) by

$$\beta^4 = \frac{3(1-\nu^2)}{(r_2\, t)^2} \tag{18}$$

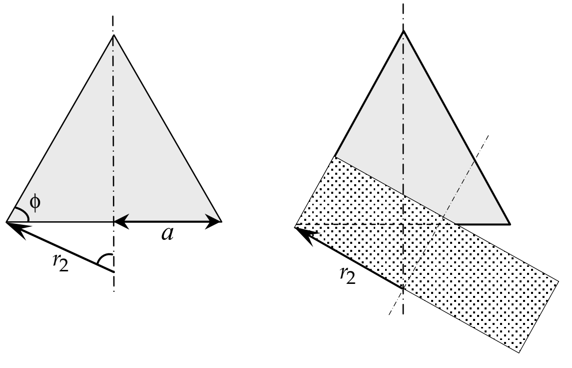

Figure 8. Definition of angle ϕ and radius r_2 (left), touching cylinder (right)

We can still use equations (8) up to (10) which we derived for circular cylinders. Said in other words, we can replace the cone with base radius a by *a* circular cylinder with larger base radius r_2 to obtain the correct differential equation for the edge zone. We show this in the right part of Figure 8. The replacing cylinder touches the cone such that the walls of the cylinder and cone coincide. As said before, we arrive at the same result which Geckeler derived by mathematical considerations.

3.2 *Edge force and torque*

The last step is to see how Eq. (12) for edge loads in a cylindrical shell changes for edge loads in a cone as shown in Figure 9, for which we refer to Figure 10. Decomposition of the force f_r yields components f_x and f_z. The component f_x yields membrane forces in x-direction, which have vanished at the upper boundary of the edge zone. The component f_z has the size

$$f_z = f_r \sin\phi \tag{19}$$

Similarly the radial displacement u_r is related to the normal displacement u_z as earlier seen in Eq. (13)

$$u_r = u_z \sin\phi \tag{20}$$

The rotation φ_x needs no adaption. As a result we have two transformations:

$$\begin{aligned} \begin{bmatrix} f_z \\ t_x \end{bmatrix} &= \begin{bmatrix} \sin\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_r \\ t_x \end{bmatrix} \\ \begin{bmatrix} u_r \\ \varphi_x \end{bmatrix} &= \begin{bmatrix} \sin\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_z \\ \varphi_x \end{bmatrix} \end{aligned} \tag{21}$$

If we apply these transformations to Eq. (12)

$$\begin{bmatrix} \sin\phi & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{D} \begin{bmatrix} \frac{1}{2\beta^3} & \frac{1}{2\beta^2} \\ \frac{1}{2\beta^2} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \sin\phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_r \\ t_x \end{bmatrix} = \begin{bmatrix} u_r \\ \varphi_x \end{bmatrix} \tag{22}$$

we obtain

$$\frac{1}{D} \begin{bmatrix} \frac{\sin^2\phi}{2\beta^3} & \frac{\sin\phi}{2\beta^2} \\ \frac{\sin\phi}{2\beta^2} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} f_r \\ t_x \end{bmatrix} = \begin{bmatrix} u_r \\ \varphi_x \end{bmatrix} \tag{23}$$

We conclude that the term $\sin\phi$ must be applied two times. It plays a role in the determination of β, and it appears in the flexibility matrix of Eq. (23).

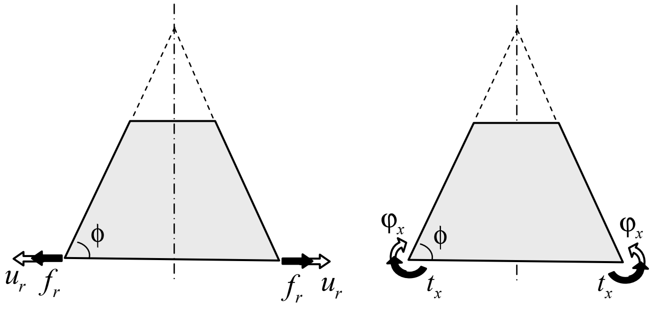

Figure 9. Edge load f_r and torque t_x, displacement u_z and rotation φ_x

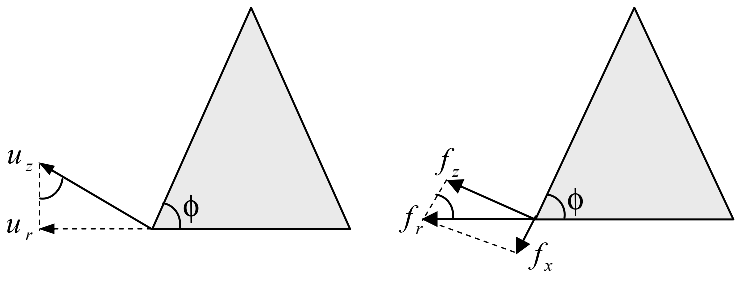

Figure 10. Decomposition of displacement u_z and force f_r

Similarly as said for the cylindrical shell, Eq. (23) facilitates satisfying the boundary equations at the edge for each loading case in which both a membrane state and edge disturbance occur in combination. After the edge force f_r and torque t_x have been determined, we apply Eq. (19) to calculate f_z from f_r and then proceed with f_z and t_x to calculate values of C and ψ from Eq. (11), after which edge zone moments and forces follow from Eq. (9).

4 Discussion of accuracy

In [3] the authors pay attention to the accuracy of the application of the edge disturbance theory of circular cylindrical shells to cones and spherical caps. They analysed a thin-walled cone as shown in Figure 9 by Finite Element Analysis. The base radius was 1000 mm, the height 1000 mm, the thickness 10 mm, Young's modulus 2.1×10^5 MPa, Poisson's ratio 0.3 and the horizontal load at the base 100 N/mm. At the top edge of the cone no boundary conditions were set. The base edge is free to roll outward horizontally, but cannot rotate. In the FE analysis 40 elements over height of the cone are applied. For the angle between the wall and the base plane they choose 60°. The maximum difference between theory and FE analysis for the dominant forces and moments appeared to be less than 1%.

A second accuracy check regarded a spherical cap. Here the authors borrowed an exact solution from Timoshenko's book *Theory of pates and shells*. The spherical cap is clamped at the edges and is loaded by a distributed homogeneous load. The angle between the wall and the base plane is 35°. The thickness radius rate is 1/30 and Poisson's ratio 1/6. The difference in the dominant bending moment appears to be about 15% and in the circumferential membrane force 5%. For design purposes this accuracy may do, for final stress checks definitely not. For very small values of ϕ between 20° and 30°, the result starts to deviate anyhow too much. Based on these examples the authors hypothesise that an angle ϕ between 45° and 90° is a safe application restriction for the thin shell edge disturbance theory.

5 Conclusions

The edge disturbance theory for rotationally symmetric shells as derived by Geckeler can be presented in a way which is more comprehensible to structural engineers. The concept of staves and rings provides insight into the physical background of the edge phenomenon and introduced approximations. The derivation is done for a cone, however, it is also applicable to spherical caps.

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