Heron's fountain 17

Finds and ideas with a surprising element similar to the playful inventions of Heron of Alexandria, after whom this journal is named

Size and shape revisited in the light of 400 years old Cavalieri principle

This fountain deals with shape and size of objects basically on arbitrary scale. We have selected for the present purpose as prime macroscopic object, the drumlin. This is a frequently occurring landform resulting from erosion. Engineers are frequently unaware of matured theorems from (far) before the digital revolution, despite being relevant for the engineering job. So, this is a first target of this fountain: explaining Cavalieri's "modern" ideas for assessment of the 3D volume or of a 2D cross sectional area of an object. Though 400 years old, Cavalieri's principle is still widely used for the objected purpose. It should be a "playful discovery" to see that it could be a handy and effective tool specifically in the drumlin case, however also for concrete engineering problems. This is targeted herein.

A sieve curve of a concrete aggregate supposedly offers but information on the size distribution and is generally at the basis of concrete standards. Concrete engineers are used to determine aggregate shape separately such as by packing experiments. As an example, the relevant IAPST (Index for Aggregate Particle Shape and Texture) test is incorporated in the ASTM standards (Hu and Stroeven, 2006). However, it is recognized already for a long time that shape and size information are mixed in a non-defined way in such approaches.

We will complete the drumlin case by introducing a new shape assessment tool. The lemniscate has been used for this purpose. It is demonstrated that the latter "blunt" tool can be transformed into an appropriate and effective "sharp" one by a simple mathematical operation: a playful discovery in the spirit of Heron of Alexandria.

1 Prime study objects

A drumlin, from the Gaelic word droimnín ("little ridge"), is an elongated whale-shaped hill formed by glacial ice acting on underlying unconsolidated till or ground moraine. Drumlins and drumlin clusters are glacial landforms which have been extensively studied. Drumlins are found in Nordic countries, but also on lower latitudes where glaciers covered areas during ice ages. Under analogous conditions, drumlins have been formed in Canada, USA and Russia. However, drumlins are also found in the Alps and on the southern hemisphere in Patagonia and even in Antarctica. The most widely accepted idea is that they were formed when the ice became overloaded with sediment. When the competence of the glacier was reduced, material was deposited in the same way that a river overloaded with sediment deposits the excess material. If there is an obstacle on the ground, this may act as a trigger point and till will build up around it. It is difficult to understand how the material could have been directly deposited in the characteristic shape of a drumlin unless the ice was still moving at the time, but it may also have been reshaped by further ice movements after it was deposited.

Drumlins can reach a kilometre or more in length, 500 m or so in width and over 50 m in height. It is common to find several drumlins grouped together. The collection of drumlins is called a swarm. In the diagram of Fig. 1, the ice was flowing from left to right. The long axis of the drumlin is the line A-B, the point of maximum width is the line C-D, and the highest point on the landform is at E. Not all drumlins will show such a distinct difference

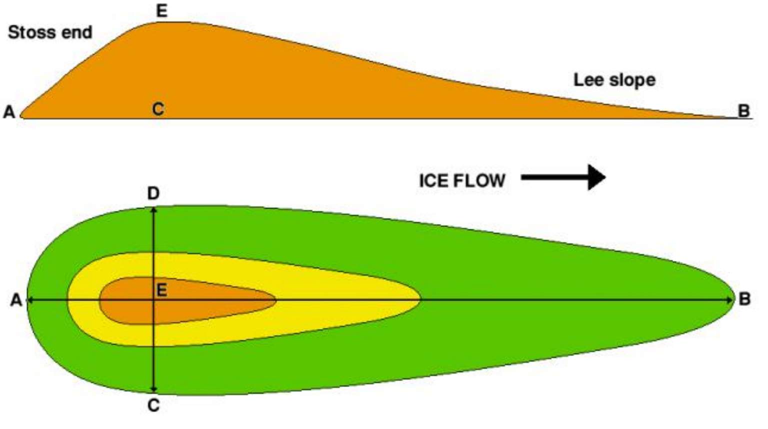

Figure 1: Plan and profile diagrams of a typical drumlin

in slope angle between the head end and lee slope, but the head end will always be the steeper of the two.

Somewhat similar deposition and erosion processes can be identified at the beach, such as demonstrated in Fig. 2, where drumlin-shaped elongated sand formations have been developed at the lee side of shells. Comparison with the drumlin swarm of Fig. 3 reveals striking similarities. Apparently, erosion processes in a preferred direction either by wind or by water (and ice) can lead to similar structure formation although on different scales.

2 Cavalieri's approach to volume and area assessment

More than half a century ago, the International Society for Stereology was founded jointly in Germany and in the USA (Weibel, 1987). The connotation "stereology" was thereby claimed and in those days defined as the science for obtaining 3D information from 2D images (sections or projections). This formed a common problem for researchers in the various material technologies but also in life sciences. As a result, it stimulated theoreticians by developing stereological concepts supporting the experimentalists. Moreover, the aforementioned researchers – among which a limited number active in concrete technology – were stimulated by contacts with colleagues from other scientific fields, so that the quality of research was promoted.

Figure 2: Deposition of sand at the lee side of shells and wind erosion created drumlin-shaped elements at the beach

With the significant developments in stereology that were realized in the following decades, also the awareness rose of long-existing important stereological concepts, *avant la lettre* so to say. For a relevant paper, also dealing with Cavalieri, see Stroeven and Hu (2006). The old roots of stereology go even back to Archimedes who developed a similar but more restricted method as Cavalieri for volume assessment. Cavalieri (1598-1647) was a fine mathematician and disciple of Galileo. In his famous book "*Geometria Indivisibilibus*" he developed concepts for area assessment by adding up lines and for volume assessment by adding up areas. Two recent books offer information on this topic to readers not yet experienced in the field of stereology (Howard and Reed, 2005), and alternatively to experts (Baddeley and Jenssen, 2006). Yet, already in 1984, Stereo (Stereo, 1984) proposed that "the unbiased estimator of volume of an arbitrary solid from parallel systematic sections that is nowadays a widely used stereological tool should be called after Cavalieri, as a tribute and recognition of the old roots of this stereological method" (Stroeven and Hu, 2006)

Cavalieri's method (also referred to as principle) for volume assessment is illustrated in Fig. 4 and 5. The most appropriate one for the present engineering problem is illustrated in Fig. 4. It requires the availability of suitable contour maps (say 3 m contours on 1:20,000 maps) (Rose and Letzer, 1975). Volume is estimated by averaging the successive areas enclosed by the contour lines and multiplying by the drumlin's height. So, $V = \overline{A}_h H$, where $\overline{A}_h$ is the average area of the horizontal serial sections and H is the height of the drumlin. For a swarm of drumlins (example shown in Fig. 3) this could even be automated. In Fig. 5 the average area of vertical serial sections is determined and multiplied by the length of the drumlin. So, $V = \overline{A}_v H$, where $\overline{A}_v$ is the average area of the vertical sections and L is the length of the drumlin.

In two different applications, either the area solely enclosed by the top contour and by the bottom contour were averaged and multiplied by the (approximate) drumlin's height (Crozier, 1975). Next, the (approximate) average cross sectional area was multiplied by the drumlin's length (Hättestrand *et al*., 2004), as shown in Fig. 5. In both cases, this can be interpreted as an approach whereby (unknowingly) use is made of the four centuries old Cavalieri principle (Stereo, 1984). Sometimes digitized images were used and slopes derived from contour maps employed for more accurately defining the base contour (Saha and Munro-Stasiuk, 2009). Also stereoscopic evaluation by aerial photography was pursued (Hättestrand *et al*., 2004). Anyhow, the area enclosed by contour lines should be estimated. This is nowadays conducted by point counting, introduced originally by Thomson (1930) and Glagolev (1931) (see, Stroeven, 1973). Nevertheless, Cavalieri provided also the solution for this 2D case by determining the average length of transects (of a line grid) and multiplying this with the tangent height perpendicular to the grid orientation.

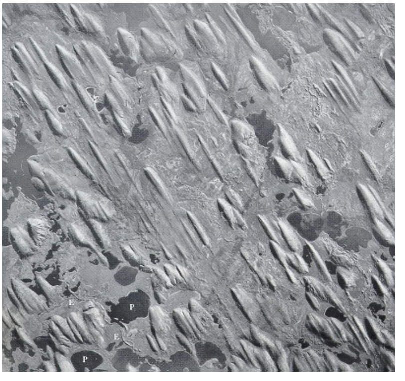

Figure 3: A swarm of Drumlins in Canada, orientation is Northeast to Southwest

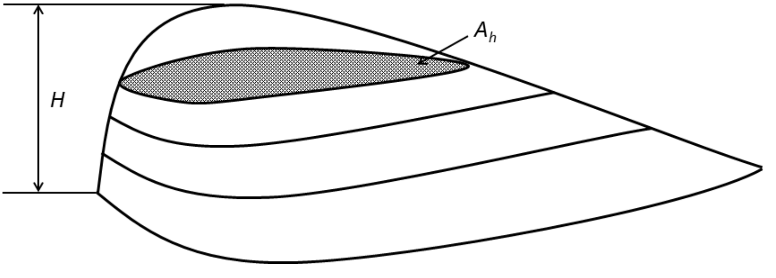

Figure 4: Serial sections in horizontal direction of a drumlin obtained from a contour map

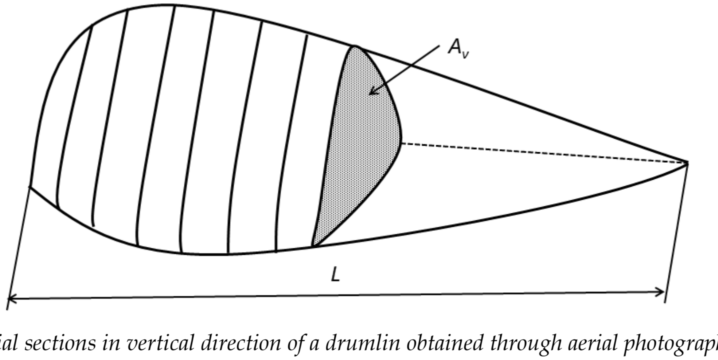

Figure 5: Serial sections in vertical direction of a drumlin obtained through aerial photography

Of course, objects of any size level could be subjected to Cavalieri's principle. The approach is common in life sciences (see, *e.g*. Gundersen *et al*., 1988), where soft tissues can be cut easily. Hard concrete is more difficult to cut, yet, this barrier is non-existing when dealing with 'compu(ter)crete' problems, *i.e.* with virtual realizations of the cementitious material. One of the more advanced methods to determine volume is developed in stereology on the basis of Cavalieri's idea (the so called Cavalieri estimator). Fig. 6 presents an application to aggregate. Orthogonal projection images are made from which the original aggregate grain is reconstructed (Stroeven and Hu, 2006; He *et al*., 2015). The Cavalieri estimator involves preparing a single representative random section. Multiplying its area A_i by the tangent height H of the particle perpendicular to the section plane yields an unbiased estimate of the particle's volume.

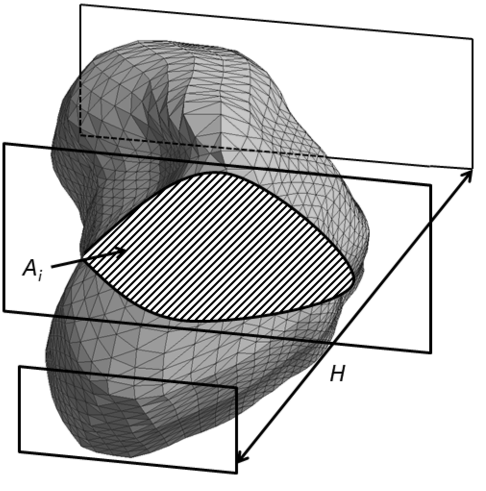

Figure 6: Concrete aggregate grain reconstructed from orthogonal images. Unbiased estimate of volume equals the area of a random section Ai times the tangent height perpendicular to the section plane, H

For the microlevel we can operate similarly. When serial sectioning and 3D reconstruction is used to reconstruct the pore network structure in virtual hardened cement paste (Ye, 2003), *local pore volume* is directly governed in a Cavalieri type of approach by the average area of successive serial sections of the *same* pore multiplied by the distance between sections. In Le (2015) and Li *et al*. (2015), a robotics type of pore delineation approach (DRaMuTS) is employed. Pore volume is governed in this case straightforwardly by local point densities among the about 10^5 points distributed uniformly random in pore space in 100 μm hardened cement paste cubes.

This point counting method is the most modern stereological method for quantitative image analysis (Stroeven, 1973). By distributing the points systematically in space, the same average information would be obtained (scatter information is modified, however). By visualizing points in planes, the link with Cavalieri becomes apparent, again! The lineal analysis for quantitative image analysis was introduced in 1898 by Rosiwal, significantly preceding the point counting method. Nowadays, we see this stereological method as a 2D version of Cavalieri's principle. Yet, it lasted 2.5 centuries before this so called "lineal analysis" was 're-invented' by Rosiwal, whereupon it finally found its way to the ASTM standards (Stroeven, 1973; Underwood, 1967; Serra, 1982) and into the concrete technology field.

3 Characterization of drumlin's shape and of other similar erosion products

Shape will be approached in this section in a "playful" way. A commonly used parameter in 3D is the surface to volume ratio and in 2D the perimeter length to area ratio. Perimeter length in sections (alternatively, surface are in 3D) is generally obtained by a random secants analysis as applied by Stroeven (1973).

Another frequently used approach to shape assessment is by standard geometric figures. The ellipse is used for the assessment of drumlin's shape (Komar, 1984), however here we will concentrate on the lemniscate, earlier used by Crozier (1975) for characterization of the drumlin's geometry. Since a drumlin is generally more slender than the lemniscate, a linear transformation of the lemniscate is proposed herein by the present authors. It will be demonstrated that such a simple transformation leads to a superior shape assessment tool. We see that as a playful discovery.

The lemniscate is given in a Cartesian system by (Bronstein and Semendjajew, 1966)

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0 \tag{1}$$

The extension in x-direction of a loop is $a\sqrt{2}$ and its width a (Fig. 7). The transformation is accomplished by $z = \frac{y}{\sqrt{b}}$. With $b > 1$, the loop's width will be reduced, so that the slenderness of the loop is made to better approximate the drumlin's shape. The width and length of the drumlin are W and L, respectively. So, $L = a\sqrt{2}$ and $W = \frac{a}{\sqrt{b}}$, whereby W and L are observations obtained in the field or derived from a contour map. The characteristics

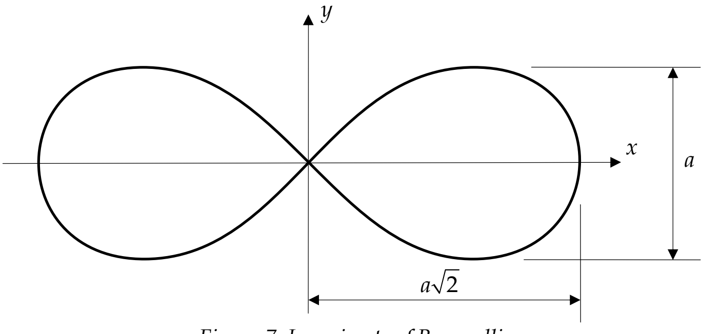

Figure 7: Lemniscate of Bernoulli

of the lemniscate are thus $a = L/\sqrt{2}$ and $b = L^2/(2W^2)$.

As an example, the parameters of a lemniscate that approximates the shape of a drumlin with a slenderness of 10 (= L/W) (Rattas and Piotrowski, 2003) and a length L_0 will be $a = L_0/\sqrt{2}$ and b = 50. Hence, the lemniscate is linearly compressed (in y-direction) by a factor $\sqrt{50} \approx 7$. The area enclosed by the original loop is $A = a^2$. So, the area enclosed by the perimeter at the drumlin's base is $\frac{a^2}{\sqrt{b}}$; for this special case $\frac{a^2}{\sqrt{50}}$. The volume of the drumlin would be one third of the area enclosed by the perimeter at the drumlin's base times drumlin height H provided it had flat surfaces ($V = \frac{H\,a^2}{3\sqrt{50}}$). H could be estimated from the contour map. If an unbiased estimate by application of the Cavalieri principle is available V_c, then $V_c \gg \frac{H\,a^2}{3\sqrt{50}}$ indicates that the drumlin's slopes get steeper towards its base.

Additionally, a parameter k defined by Chorley (1959), $k = \frac{\pi L^2}{4A}$, is in use for characterizing the shape of a drumlin's base by way of a lemniscate. This yields k = 1 for a circle and k = a/b for an ellipse, with a and b as principal axes and $a > b$. Hence, for the aforementioned example k = 1.6 for the lemniscate. For the drumlin, at the same length, a smaller area will be obtained, increasing the value of k. Values ranging from 2 and 10 are found (Rattas and Piotrowski, 2003; Gravenor, 1974). *This does not lead to proper matching of shapes, of course.* The approach by the present authors' transformed lemniscate would lead to $k = \frac{\pi\sqrt{2}}{4}\frac{L}{W}$. So, this is directly correlated with the slenderness of the drumlin. For the example case of a drumlin with a slenderness of 10, k = 11. This is close to the value for an ellipse with the same slenderness as the drumlin (k = 10). The compressed lemniscate *would be superior in shape approximation* as may be concluded from Fig. 8. This was the target of this section, *i.e.*, shape simulation and quantification through similarity with an analytically described geometric figure readily obtained by a simple mathematical operation.

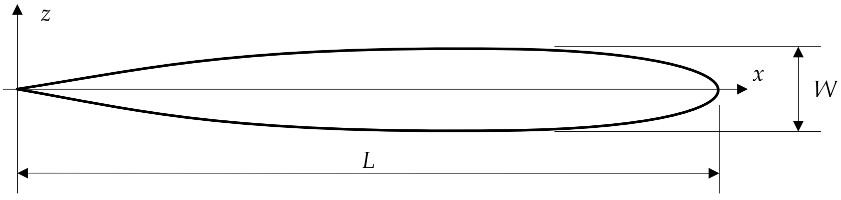

Figure 8: Shape simulation by a compressed lemniscate with a slenderness of 10. Compare with the characteristic shape presented in Fig. 1 and with that of the drumlins in the swarm of Fig. 3

4 Discussion and conclusions

The 400 years old ideas by Cavalieri are still directly applicable for volume and areal assessment, not just in the case of large-scale landforms or other erosion products, but also to, *e.g.*, aggregate grains. Concrete pores are at the other end of the microscale. Even here Cavalieri's ideas can find profitable use. Some methods developed in the past for this and other quantitative morphological purposes can be considered influenced by Cavalieri's method or are his ideas dressed differently. Since size (volume is a size measure) and shape cannot unambiguously be separated, a playful approach to 2D shape assessment was included. Traditionally, shape is characterized by similitude to elementary figures, like circle, ellipse, square, rectangle or triangle. Here the lemniscate was linearly transformed for characterizing the shape of drumlins; a simple mathematical operation, leading to superior shape simulation: a real playful discovery!

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