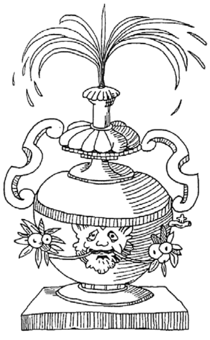
Heron's fountain 16



Finds and ideas with a surprising element similar to the playful inventions of Heron of Alexandria, after whom this journal is named

The sectio aurea of structural engineers

Consider two parallel walls (Fig. 1). Spanning between the walls are large beams. Spanning between the large beams are smaller beams. Spanning between the smaller beams are even smaller beams or planks. All beams are simply supported and loaded in bending. In this fountain the following question is answered. What is the optimal spacing of the beams?

In the beam grid we can recognise a timber floor from the Middle Ages (in Dutch: moer- en kinderbinten). Modern applications are window frames, timber frame buildings, orthotropic steel bridges, airplane fuselages, ship hulls and the new generation of space vehicles that is being developed in the USA.

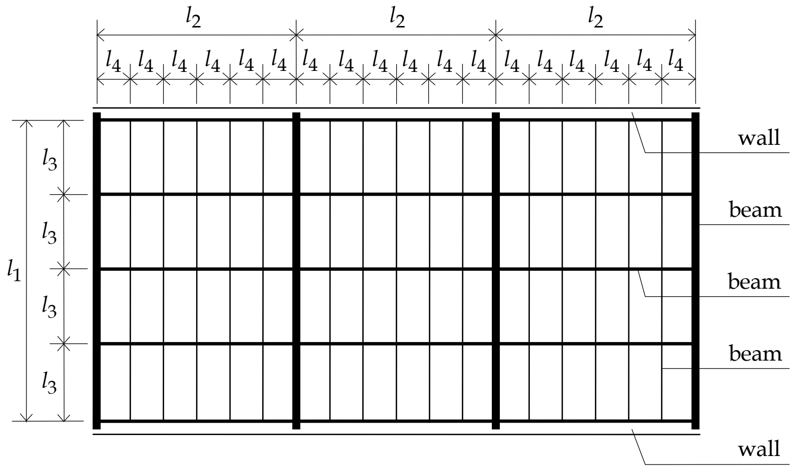


Figure 1. A grid of simply supported beams loaded in bending

The beams have different lengths. These lengths are numbered from 1 to m–1 (Fig. 1). The largest moment in a beam with a length l_i is approximated by

$$M_i = \tfrac{1}{8} p\, l_{i+1}\, l_i^2 \,, \tag{1}$$

where p is a distributed load [kN/m²]. Self-weight of the beams is added to the distributed load. It is assumed that the beam cross-sections are rectangular with a width b and a depth h. Beams of different length have different cross-section dimensions but the aspect ratio is the same for the cross-sections of all beams.

$$b_i = \alpha h_i \tag{2}$$

The largest stress in each beam is equal to the allowable stress f.

$$f = \frac{M_i}{\tfrac{1}{6} b_i h_i^2} \tag{3}$$

The number of beams of one length is

$$n_i = \frac{A}{l_i\, l_{i+1}} \tag{4}$$

where A is the area covered by the beam grid. The volume of all beams together is

$$V = \sum_{i=1}^{m-1} b_i\, h_i\, l_i\, n_i \,. \tag{5}$$

It may seem that edge beams are neglected. However, edge beams carry half the surface of normal beams and have half the width of normal beams. Therefore, two edge beams are equivalent to one normal beam, which is perfectly described by the equations.
The material volume is minimal with respect to the beam spacing.

$$\frac{\partial V}{\partial l_i} = 0 \qquad i = 2 \text{ to } m-1 \tag{6}$$

Note that the span l_1 and the smallest beam spacing l_m are not variables. Substitution of Eq. 1 to 5 in Eq. 6 gives a surprisingly simple result.

$$l_{i+1} = 64 \frac{l_i^5}{l_{i-1}^4} \qquad i = 2 \text{ to } m-1 \tag{7}$$

Most variables have dropped out of the equation such as the aspect ratio α, the allowable stress f and the distributed load p.
Example 1: if l_1 = 16 m and l_2 = 3 m then l_3 = 0.24 m, l_4 = 0.0006 m, l_5 = 0.000000000001 m, etc. In a realistic application we interrupt the sequence after 2 and replace the number 3 beams by panels.

Example 2: if l_1 = 16 m and l_2 = 4 m then l_3 = 1 m, l_4 = 0.25 m, l_5 = 0.0625 m, etc. In a realistic application we cut the sequence after 3 and replace the number 4 beams by floorboard. Alternatively, we can cut after 4 and replace the number 5 beams by roofing tiles.
Example 3: if l_1 = 16 m and l_2 = 5 m than l_3 = 3.05 m, l_4 = 27.1 m, l_5 = 10795211 m, etc. This last example is not realistic after beams 2 because beams 3 have a spacing larger than their lengths. The examples show that l_2 has a large influence on the optimal grid. Clearly, we prefer to select l_2 such that it provides the right l_m for supporting the product that closes the surface.

We can also start with the smallest beam spacing l_m and calculate the larger beam spacings. To this end Eq. 7 is rewritten.

$$\frac{l_i}{l_{i-1}} = \left(\frac{1}{64} \frac{l_{i+1}}{l_i} \right)^{\frac{1}{4}} \tag{8}$$

Starting with $\frac{l_{i+1}}{l_i}$ = 0.05 the sequence is, $\frac{l_i}{l_{i-1}}$ = 0.167, 0.226, 0.244, 0.248, 0.249, etc. It converges quickly to $\frac{1}{4}$ for any positive starting value. It can be concluded that approximately $l_{i+1} = \frac{1}{4} l_i$. The smallest beam spacing can be rounded to the required value (Fig. 2).

The table on the next page shows the results of a detailed study. Every beam grid has a span l_1 and a spacing of the smallest beams l_m . The ratio of these numbers is shown in the first column. From this ratio follows the optimal number of beam lengths shown in column 2. From this follows the spacing of the first beams shown in column 3. From this follows the material volume shown in column 4.

The analysis has been repeated for grids of continuous beams instead of simply supported beams. Except for the material volume in the table, the results for these grids are exactly the same.

Architects have their sectio aurea or golden ratio of 1 : $\frac{1}{2}(1-\sqrt{5})$. It appears that structural engineers also have a sectio aurea. It is simply 1 : 4 .

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Table. Characteristics of optimal beam grids

spacing of the smallest beams	number of beam lengths	spacing of the largest beams	total material volume
$1.00 < \frac{l_1}{l_2} < 6.53$	1	$1.000 < \frac{l_1}{l_2} < 6.531$	$0.825 < \frac{V}{A l_1 \sqrt[3]{\alpha}}(\frac{f}{p})^{\frac{2}{3}} < 1.543$
$6.53 < \frac{l_1}{l_3} < 27.1$	2	$3.344 < \frac{l_1}{l_2} < 4.445$	$1.543 < \frac{V}{A l_1 \sqrt[3]{\alpha}}(\frac{f}{p})^{\frac{2}{3}} < 1.696$
$27.1 < \frac{l_1}{l_4} < 109$	3	$3.840 < \frac{l_1}{l_2} < 4.104$	$1.696 < \frac{V}{A l_1 \sqrt[3]{\alpha}}(\frac{f}{p})^{\frac{2}{3}} < 1.734$
$109 < \frac{l_1}{l_5} < 438$	4	$3.960 < \frac{l_1}{l_2} < 4.025$	$1.734 < \frac{V}{A l_1 \sqrt[3]{\alpha}}(\frac{f}{p})^{\frac{2}{3}} < 1.744$
$438 < \frac{l_1}{l_6} < 1753$	5	$3.990 < \frac{l_1}{l_2} < 4.006$	$1.744 < \frac{V}{A l_1 \sqrt[3]{\alpha}}(\frac{f}{p})^{\frac{2}{3}} < 1.746$

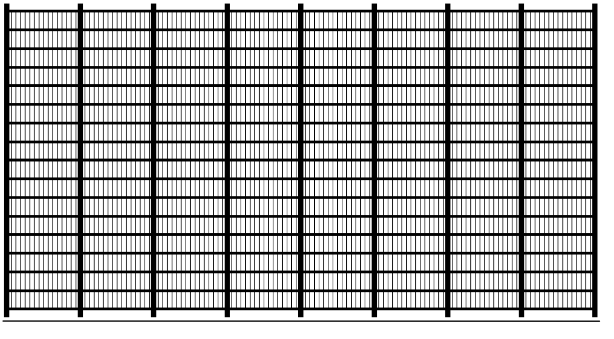

Figure 2. Optimal beam grid proportions 1: 4