Computation of reinforcement for solid concrete

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Reinforcement in a concrete structure is often determined based on linear elastic stresses. This paper considers computation of the required reinforcement when these stresses have been determined by the finite element method with volume elements. Included are both tension reinforcement and compression reinforcement, multiple load combinations and crack control in the serviceability limit state. Results are presented of seventeen stress state examples.

Key words: Reinforcement design, thee-dimensional stresses, optimisation, FEM, postprocessing

1 Introduction

Many computer programs for structural analysis have post processing functionality for designing reinforcement and performing code compliance checks. For example the moments and normal forces computed with shell elements can be used to determine the required reinforcement based on the Eurocode design rules [1]. However, for finite element models containing volume elements, reinforcement design rules do not exist. Software companies that are developing structural analysis programs are in the process of extending the program capabilities with volume elements. Consequently, also the algorithms for computing reinforcement requirements need to be extended for use with volume elements.

Already in 1983, Smirnov [2] pointed out the importance of this problem for design of reinforced concrete in hydroelectric power plants. Unfortunately, the design rule that he proposed in his paper is incorrect. Following his assumptions he should have arrived at Eq. 4 of this paper. Kamezawa et al. [3] proposed five design rules for three-dimensional reinforcement design. Among these is Eq. 4 of this paper while the other four design rules used are theoretically incorrect. The design rules were tested on an example structure by applying reinforcement according to each rule and performing nonlinear finite element analyses up to failure. Their best performing design rule produces the same reinforcement as Eqs 13-16 of this paper for their example structure. However, in another structure it can significantly overestimate the required amount of reinforcement. Foster et al. [4] derived the correct design rules for the interior solution, which are Eqs 13-17 in this paper. However, they rely on Mohr's circle and graphs to determine which of these rules to use. This makes their approach not suitable for computer implementation.

In this paper an analytical approach and a numerical approach are followed. The analytical approach yields a complete set of design rules for determining tension reinforcement for the ultimate limit state. The set includes the rules that have been derived by Foster [4]. It also includes the rules that are commonly used for design of reinforced concrete in a plane stress state [1]. The numerical approach has the advantage that in addition multiple load combinations, compression reinforcement and crack control can be included in the computation.

Reinforced concrete often has many small cracks that developed during curing of the material as a result of the interaction of the shrinking concrete and the reinforcing bars. Therefore, we cannot rely on concrete having tensile strength. The reinforcement needs to be computed such that the concrete principal stresses are smaller than zero. This is fulfilled when the first concrete principal stress is smaller than zero.

$$\sigma_{c1} \leq 0$$

Also, the concrete compressive stresses need to fulfil a condition. The Mohr-Coulomb yield contour is often used as a conservative condition for preventing concrete failure.

$$\frac{\sigma_{c3}}{\sigma_c} + \frac{\sigma_{c1}}{\sigma_t} \leq 1$$

In this σ_c is the uniaxial concrete compressive strength (negative number) and σ_t is the concrete tensile strength. Here, the tensile strength is larger than zero because it is an

average value instead of a local value. If the concrete stresses are too large, compression reinforcement or confinement reinforcement can be a solution.
According to the lower bound theorem of plasticity theory [5, 6] any force flow that is in equilibrium and fulfils the strength conditions of the materials provides a safe solution for the carrying capacity of the structure. Thus, designing reinforcement for the ultimate limit state is an optimisation problem: Minimise the amount of reinforcement with the above mentioned conditions on the concrete principal stresses.

In addition, the crack width needs to be limited for load combinations related to the serviceability limit state.

$$w \leq w_{\mathrm{max}}$$

This condition is imposed for aesthetics and to prevent corrosion of the reinforcing steel. Often, this condition alone determines the required reinforcement ratios.

In reinforced concrete beam design it is customary to include at least a minimum reinforcement. This is to ensure ductile failure and distributed cracking. However, in many situations the minimum reinforcement requirements result in much more reinforcement than reasonable. Therefore, in this paper it is not considered. Of course, a design engineer can decide to apply at least minimum reinforcement according to the governing code of practice.

In Appendix 1 a short summary is given of the Theory of Elasticity to explain the notations and definitions used in this paper.

2 Equilibrium of forces

Figure 1 shows the stresses on a small material cube. The stress values are known since they are computed by a finite element program. Figure 2 shows part of this cube with a crack and a reinforcing bar. We assume that the reinforcing bars are directed in the x, y and z directions. In Figure 2 only the reinforcing bar in the x direction is shown. The reinforcement stress σ_y in a crack needs to be in equilibrium with the stresses on the cube faces. We assume that the normal stresses and shear stresses on the crack face are zero.

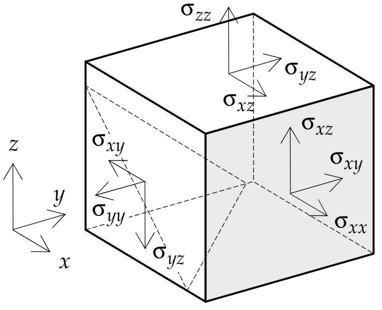

Figure 1. Stresses on a small material cube

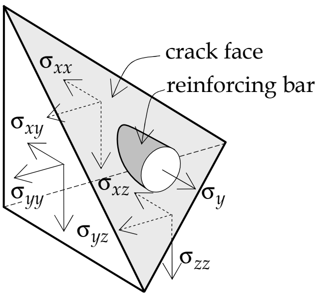

Figure 2. Equilibrium of a cracked cube part

The equilibrium equations of the cracked cube part are

$$\begin{aligned}
\sigma_y A\cos\alpha\rho_x &= \sigma_{xx}A\cos\alpha + \sigma_{xy}A\cos\beta + \sigma_{xz}A\cos\gamma \\
\sigma_y A\cos\beta\rho_y &= \sigma_{xy}A\cos\alpha + \sigma_{yy}A\cos\beta + \sigma_{yz}A\cos\gamma \\
\sigma_y A\cos\gamma\rho_z &= \sigma_{xz}A\cos\alpha + \sigma_{yz}A\cos\beta + \sigma_{zz}A\cos\gamma
\end{aligned} \tag{1}$$

Where ρ_x, ρ_y, ρ_z are the reinforcement ratios in the *x*, *y* and *z* directions. *A* is the crack face area. α, β And γ are the angles of the crack face normal vector. In the derivation of the equations the geometrical relations shown in Figure 3 have been used.

Eqs (1) can be rewritten as

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}\sigma_{xx}-\rho_x\sigma_y & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy}-\rho_y\sigma_y & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\rho_z\sigma_y\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{bmatrix} \tag{2}$$

This matrix will be referred to as the concrete stress tensor. The concrete principal stresses σ_{c1} , σ_{c2} and σ_{c3} are the eigenvalues of this matrix. Non trivial solutions of Eqs (2) can be found when the determinant I_{c3} of the matrix is zero.

$$I_{c3} = \sigma_{cx}\sigma_{cy}\sigma_{cz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{cx}\sigma_{yz}^2 - \sigma_{cy}\sigma_{xz}^2 - \sigma_{cz}\sigma_{xy}^2 = 0 \tag{3}$$

where

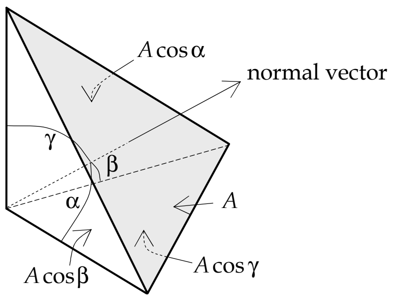

Figure 3. Surface areas of the cracked cube part

$$\sigma_{cx} = \sigma_{xx} - \rho_x \sigma_y$$

$$\sigma_{cy} = \sigma_{yy} - \rho_y \sigma_y$$

$$\sigma_{cz} = \sigma_{zz} - \rho_z \sigma_y \,.$$

The problem can be visualised in a graph (Fig. 4). The axis of this graph represent ρ_x, ρ_y and ρ_z. The condition $I_{c3} = 0$ is shown as a surface. We are looking for the smallest possible value of $\rho_x + \rho_y + \rho_z$ on this surface. The shape of the surface depends on the linear elastic stress tensor and the steel stress σ_y. Not only interior solutions but also edge and corner solutions are possible (Fig. 4).

3 Principal reinforcement

Suppose that we select the following reinforcement

$$\rho_x = \frac{\sigma_1}{\sigma_y}, \quad \rho_y = \frac{\sigma_1}{\sigma_y}, \quad \rho_z = \frac{\sigma_1}{\sigma_y}\,. \tag{4}$$

σ_1 is the largest eigenvalue of the linear elastic stress tensor. Substitution of Eqs (4) in Eqs (2) gives

$$\begin{bmatrix} \sigma_{xx} - \sigma_1 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_1 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_1 \end{bmatrix}.$$

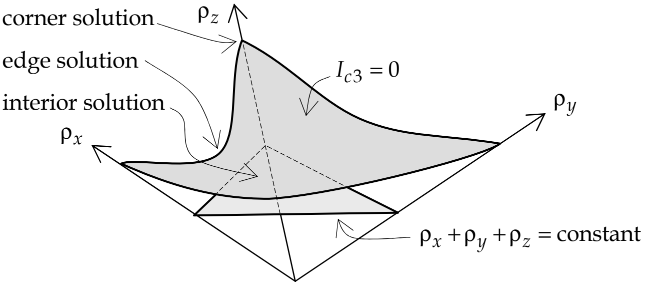

Figure 4. Conceptual presentation of the optimisation problem

Its determinant I_{c3} is zero because this is how the eigenvalue is derived in the first place. It can be shown that one of the eigenvalues of the concrete stress tensor is zero and the other two eigenvalues are smaller than or equal to zero. Therefore, the reinforcement proposed in Eq. (4) is suitable. The crack direction $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ will be equal to the first principal direction of the linear elastic stress tensor. Therefore, the crack direction in the ultimate limit state is the same as the crack direction in the serviceability limit state.

An advantage of this reinforcement is that few additional cracks will form in a material cube when the load increases towards the ultimate load. This might be beneficial to the durability of the structure. However, less reinforcement is required when we accept that the cracks in the ultimate limit state will be different from the initial cracks. Often the reinforcement can be reduced to almost one third when the reinforcement ratios are optimised.

4 Reinforcement formulas

Corner solutions

Assuming reinforcement in one direction only, the following formulas can be derived for the required amount of reinforcement ($I_{c3} = 0$).

$$\rho_x = 0, \quad \rho_y = 0, \quad \rho_z = \frac{I_3}{\sigma_y(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)} \tag{5}$$

$$\rho_x = 0, \quad \rho_y = \frac{I_3}{\sigma_y(\sigma_{xx}\sigma_{zz} - \sigma_{xz}^2)}, \quad \rho_z = 0 \tag{6}$$

$$\rho_x = \frac{I_3}{\sigma_y(\sigma_{yy}\sigma_{zz} - \sigma_{yz}^2)}, \quad \rho_y = 0, \quad \rho_z = 0 \tag{7}$$

where, I_3 is the determinant of the linear elastic stress tensor (Appendix 1). For plane stress, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, Eqs (7) reduces to

$$\rho_x = \frac{\sigma_{xx}}{\sigma_y} - \frac{\sigma_{xy}^2}{\sigma_y\sigma_{yy}}, \quad \rho_y = 0, \quad \rho_z = 0, \tag{8}$$

which is commonly used in reinforcement design of concrete walls [1].
Also the crack directions can be derived by substitution of the reinforcement ratios in Eqs (2). For example, for Eqs (7) the result is

$$\cos\alpha = \frac{\sigma_{yy}\sigma_{zz} - \sigma_{yz}^2}{l}, \quad \cos\beta = \frac{\sigma_{xz}\sigma_{yz} - \sigma_{zz}\sigma_{xy}}{l}, \quad \cos\gamma = \frac{\sigma_{xy}\sigma_{yz} - \sigma_{yy}\sigma_{xz}}{l},$$

$$l = (\sigma_{yy}\sigma_{zz} - \sigma_{yz}^2)^2 + (\sigma_{xz}\sigma_{yz} - \sigma_{zz}\sigma_{xy})^2 + (\sigma_{xy}\sigma_{yz} - \sigma_{yy}\sigma_{xz})^2 .$$

Edge solutions

Assuming reinforcement in two directions only, the following formulas can be derived for the smallest amount of reinforcement ($I_{c3} = 0$ and $\frac{d(\rho_y + \rho_z)}{d\rho_y} = 0$, etc.).

$$\rho_x = 0, \quad \rho_y = \frac{\sigma_{yy}}{\sigma_y} - \frac{\sigma_{xy}^2}{\sigma_y\sigma_{xx}} \pm \left(\frac{\sigma_{xz}\sigma_{xy}}{\sigma_y\sigma_{xx}} - \frac{\sigma_{yz}}{\sigma_y}\right), \quad \rho_z = \frac{\sigma_{zz}}{\sigma_y} - \frac{\sigma_{xz}^2}{\sigma_y\sigma_{xx}} \pm \left(\frac{\sigma_{xz}\sigma_{xy}}{\sigma_y\sigma_{xx}} - \frac{\sigma_{yz}}{\sigma_y}\right) \tag{9}$$

$$\rho_x = \frac{\sigma_{xx}}{\sigma_y} - \frac{\sigma_{xy}^2}{\sigma_y\sigma_{yy}} \pm \left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_y\sigma_{yy}} - \frac{\sigma_{xz}}{\sigma_y}\right), \quad \rho_y = 0, \quad \rho_z = \frac{\sigma_{zz}}{\sigma_y} - \frac{\sigma_{yz}^2}{\sigma_y\sigma_{yy}} \pm \left(\frac{\sigma_{yz}\sigma_{xy}}{\sigma_y\sigma_{yy}} - \frac{\sigma_{xz}}{\sigma_y}\right) \tag{10}$$

$$\rho_x = \frac{\sigma_{xx}}{\sigma_y} - \frac{\sigma_{xz}^2}{\sigma_y\sigma_{zz}} \pm \left(\frac{\sigma_{xz}\sigma_{yz}}{\sigma_y\sigma_{zz}} - \frac{\sigma_{xy}}{\sigma_y}\right), \quad \rho_y = \frac{\sigma_{yy}}{\sigma_y} - \frac{\sigma_{yz}^2}{\sigma_y\sigma_{zz}} \pm \left(\frac{\sigma_{xz}\sigma_{yz}}{\sigma_y\sigma_{zz}} - \frac{\sigma_{xy}}{\sigma_y}\right), \quad \rho_z = 0 \tag{11}$$

For plane stress, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, Eqs (10) reduce to Eqs (8) and Eqs (11) reduce to

$$\rho_x = \frac{\sigma_{xx} \mp \sigma_{xy}}{\sigma_y}, \quad \rho_y = \frac{\sigma_{xx} \mp \sigma_{xy}}{\sigma_y}, \quad \rho_z = 0 \tag{12}$$

which is commonly used in reinforcement design of concrete walls too [1].

For Eqs (11) the crack directions are

$$\cos\alpha = \frac{\pm\sigma_{zz}}{\sqrt{2\sigma_{zz}^2 + (\sigma_{yz} \mp \sigma_{xz})^2}}, \quad \cos\beta = \frac{-\sigma_{zz}}{\sqrt{2\sigma_{zz}^2 + (\sigma_{yz} \mp \sigma_{xz})^2}}, \quad \cos\gamma = \frac{\sigma_{yz} \mp \sigma_{xz}}{\sqrt{2\sigma_{zz}^2 + (\sigma_{yz} \mp \sigma_{xz})^2}}.$$

Interior solutions

For reinforcement in three directions the following formulas can be derived for the smallest amount of reinforcement ($I_{c3} = 0$ and $\frac{d(\rho_x + \rho_y + \rho_z)}{d\rho_y} = 0$ and $\frac{d(\rho_x + \rho_y + \rho_z)}{d\rho_z} = 0$).

$$\rho_x = \frac{\sigma_{xx} + \sigma_{xy} + \sigma_{xz}}{\sigma_y}, \quad \rho_y = \frac{\sigma_{yy} + \sigma_{xy} + \sigma_{yz}}{\sigma_y}, \quad \rho_z = \frac{\sigma_{zz} + \sigma_{xz} + \sigma_{yz}}{\sigma_y} \tag{13}$$

$$\rho_x = \frac{\sigma_{xx} + \sigma_{xy} - \sigma_{xz}}{\sigma_y}, \quad \rho_y = \frac{\sigma_{yy} + \sigma_{xy} - \sigma_{yz}}{\sigma_y}, \quad \rho_z = \frac{\sigma_{zz} - \sigma_{xz} - \sigma_{yz}}{\sigma_y} \tag{14}$$

$$\rho_x = \frac{\sigma_{xx} - \sigma_{xy} - \sigma_{xz}}{\sigma_y}, \quad \rho_y = \frac{\sigma_{yy} - \sigma_{xy} + \sigma_{yz}}{\sigma_y}, \quad \rho_z = \frac{\sigma_{zz} - \sigma_{xz} + \sigma_{yz}}{\sigma_y} \tag{15}$$

$$\rho_x = \frac{\sigma_{xx} - \sigma_{xy} + \sigma_{xz}}{\sigma_y}, \quad \rho_y = \frac{\sigma_{yy} - \sigma_{xy} - \sigma_{yz}}{\sigma_y}, \quad \rho_z = \frac{\sigma_{zz} + \sigma_{xz} - \sigma_{yz}}{\sigma_y} \tag{16}$$

$$\rho_x = \frac{\sigma_{xx}}{\sigma_y} - \frac{\sigma_{xy}\sigma_{xz}}{\sigma_y\sigma_{yz}}, \qquad \rho_y = \frac{\sigma_{yy}}{\sigma_y} - \frac{\sigma_{xy}\sigma_{yz}}{\sigma_y\sigma_{xz}}, \qquad \rho_z = \frac{\sigma_{zz}}{\sigma_y} - \frac{\sigma_{xz}\sigma_{yz}}{\sigma_y\sigma_{xy}} \tag{17}$$

For Eqs (16) the crack directions are

$$\cos\alpha = \frac{1}{\sqrt{3}}, \quad \cos\beta = \frac{-1}{\sqrt{3}}, \quad \cos\gamma = \frac{1}{\sqrt{3}}.$$

For Eqs (17) the crack direction is indeterminate but perpendicular to vector

$$(\sigma_{xy}\sigma_{xz},\ \sigma_{xy}\sigma_{yz},\ \sigma_{xz}\sigma_{yz}),$$

which apparently is the direction of the concrete compressive stress.

In this section, 11 sets of formulas are presented as potential solutions of the least amount of reinforcement. For a particular stress state most of these solutions are invalid. The optimal reinforcement is either $\rho_x = \rho_y = \rho_z = 0$ or the result of one (or more) of the valid solutions. It is not attempted to specify the stress ranges for which a particular set provides the optimum. This is expected to produce very large and therefore impractical results. In Section 5, a method is proposed to test the validity of a potential reinforcement solution for a particular stress state.

The formulas consider only one stress state, therefore only one load combination. In general, for multiple load combinations, the real minimum is not predicted by any of these sets of formulas. In Section 6, a method is proposed to compute the least amount of reinforcement for multiple load combinations.

5 Testing a solution

The validity of the formulas in the previous section depends on the actual stress state. A first check is that the reinforcement ratios need to be larger than or equal to zero. It is possible to further test a formula result by computing the concrete principal stresses and checking whether these are smaller than or equal to zero. However, the computation time for this can be large because computing eigenvalues involves finding the roots of a third order polynomial. Moreover, this needs to be repeated for all sets of formulas, for all load combinations and all integration points of a finite element model. On the other hand, the invariants of the concrete stress tensor can be computed faster.

$$
\begin{aligned}
I_{c1} &= \sigma_{cx} + \sigma_{cy} + \sigma_{cz} \\
I_{c2} &= \sigma_{cx}\sigma_{cy} + \sigma_{cy}\sigma_{cz} + \sigma_{cz}\sigma_{cx} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2 \\
I_{c3} &= \sigma_{cx}\sigma_{cy}\sigma_{cz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{cx}\sigma_{yz}^2 - \sigma_{cy}\sigma_{xz}^2 - \sigma_{cz}\sigma_{xy}^2
\end{aligned}
\tag{18}
$$

The condition $\sigma_{c1} \leq 0$ is equivalent (necessary and sufficient) to

$$I_{c1} \leq 0 \tag{19a}$$

$$I_{c2} \geq 0 \tag{19b}$$

$$I_{c3} \leq 0 \tag{19c}$$

The "necessary" proof is straight forward by substitution of the principal stresses in Eq. (A4).

The "sufficient" proof follows a reductio ad absurdum. Suppose that one or all of the principal stresses is larger than zero. Then from Eq. (33) it follows that $I_{c3} > 0$ which is inconsistent with Eq. (19c). Suppose that two principal concrete stresses are larger than zero, for example $\sigma_{c1} > 0$ and $\sigma_{c2} > 0$. From Eq. (19a) and (19b) it follows that $\sigma_{c1} + \sigma_{c2} + \sigma_{c3} \leq 0$ and $\sigma_{c1}\sigma_{c2} + \sigma_{c2}\sigma_{c3} + \sigma_{c3}\sigma_{c1} \geq 0$. So $\sigma_{c3} \leq -\sigma_{c1} - \sigma_{c2} = -\frac{(\sigma_{c1} + \sigma_{c2})^2}{\sigma_{c1} + \sigma_{c2}} = \frac{-\sigma_{c1}^2 - 2\sigma_{c1}\sigma_{c2} - \sigma_{c2}^2}{\sigma_{c1} + \sigma_{c2}}$ and $\sigma_{c3} \geq \frac{-\sigma_{c1}\sigma_{c2}}{\sigma_{c1} + \sigma_{c2}}$. The latter two conditions are inconsistent too. Q.E.D.

6 Compression reinforcement

If $\sigma_{c1} = 0$ the other principal concrete stresses can be computed by

$$\sigma_{c2} = \tfrac{1}{2} I_{c1} + \sqrt{(\tfrac{1}{2} I_{c1})^2 - I_{c2}}$$
$$\sigma_{c3} = \tfrac{1}{2} I_{c1} - \sqrt{(\tfrac{1}{2} I_{c1})^2 - I_{c2}}$$

This can be derived by solving σ_2 and σ_3 from Eqs (33). When the concrete stresses are too large (in absolute sense) than compression reinforcement and confinement reinforcement can be used. The objective is the same as for tension reinforcement; minimize $\rho_x + \rho_y + \rho_z$.

The Mohr-Coulomb constraint to fulfil is $\frac{\sigma_{c3}}{\sigma_c} + \frac{\sigma_{c1}}{\sigma_t} \leq 1$. The equilibrium equations are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} - \rho_x \sigma_{sx} - \sigma_{ci} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \rho_y \sigma_{sy} - \sigma_{ci} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \rho_z \sigma_{sz} - \sigma_{ci} \end{bmatrix} \begin{bmatrix} \cos \alpha_i \\ \cos \beta_i \\ \cos \gamma_i \end{bmatrix} \quad i = 1, 2, 3 \qquad (20)$$

Each of the steel stresses σ_{sx}, σ_{sy}, σ_{sz} can be negative or positive. The problem is too complicated for analytical solution. A numerical implementation is shown in Appendix 2.

For very large reinforcement ratios the concrete true stresses are significantly larger than the average concrete stresses in Eq. 2 and Eq. 20. The following adjustments can be considered to obtain the concrete true stress tensor. However, in this paper, small reinforcement ratios are assumed and Eq. 20 is used.

$$\begin{bmatrix} \frac{\sigma_{xx} - \rho_x \sigma_{sx}}{1-\rho_x} & \sigma_{xy} \frac{1}{2}(\frac{1}{1-\rho_x} + \frac{1}{1-\rho_y}) & \sigma_{xz} \frac{1}{2}(\frac{1}{1-\rho_x} + \frac{1}{1-\rho_z}) \\ \sigma_{xy} \frac{1}{2}(\frac{1}{1-\rho_x} + \frac{1}{1-\rho_y}) & \frac{\sigma_{yy} - \rho_y \sigma_{sy}}{1-\rho_y} & \sigma_{yz} \frac{1}{2}(\frac{1}{1-\rho_y} + \frac{1}{1-\rho_z}) \\ \sigma_{xz} \frac{1}{2}(\frac{1}{1-\rho_x} + \frac{1}{1-\rho_z}) & \sigma_{yz} \frac{1}{2}(\frac{1}{1-\rho_y} + \frac{1}{1-\rho_z}) & \frac{\sigma_{zz} - \rho_z \sigma_{sz}}{1-\rho_z} \end{bmatrix} \tag{21}$$

7 Crack control

Crack width is important for load combinations related to the serviceability limit state. The crack occurs perpendicular to the first principal direction and sometimes also perpendicular to the second and third principal directions. When the load increases the crack can grow in a different direction. This is often referred to as crack rotation. Crack rotation can already be significant in the serviceability limit state.

The linear elastic strains computed by a finite element analysis could be used for determining the crack width. However, these strains would not be very accurate because they strongly depend on Young's modulus of cracked reinforced concrete which can only be estimated. On the other hand, the stresses do not depend on Young's modulus[1]. Therefore, the computation of crack widths starts from the stresses. In essence, the adopted equations are part of the Modified Compression Field Theory [7] simplified for the serviceability limit state and extended for three dimensional analysis.

Eq. (20) can be rewritten to.

[1] Except for temperature loading and foundation settlements in statically indetermined structures. For these cases an accurate estimate of Young's modulus of cracked reinforced concrete needs be used in the linear elastic analysis.

$$
\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = P \begin{bmatrix} \sigma_{c1} & 0 & 0 \\ 0 & \sigma_{c2} & 0 \\ 0 & 0 & \sigma_{c3} \end{bmatrix} P^{-1} + \begin{bmatrix} \rho_x \sigma_{sx} \\ \rho_y \sigma_{sy} \\ \rho_z \sigma_{sz} \end{bmatrix} \tag{22}
$$

where σ_{c1}, σ_{c2}, σ_{c3} are the concrete principal stresses and

$$
P = \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \cos\gamma_1 & \cos\gamma_2 & \cos\gamma_3 \end{bmatrix}. \tag{23}
$$

The columns in P are the vectors of the concrete principal directions. Note that in general these principal directions are not the same as the linear elastic principal directions. The principal direction vectors are perpendicular, therefore $P^{-1} = P^{\mathrm{T}}$. This can be proved by showing that $P^{\mathrm{T}}P = PP^{\mathrm{T}} = I$.

Since yielding is supposed not to occur in the serviceability limit state, the constitutive relations for the reinforcing bars are linear elastic. The constitutive relation for compressed concrete is approximated as linear elastic in the principal directions. Poisson's ratio is set to zero. The constitutive relation for tensioned concrete is

$$
\sigma_{ci} = \frac{\sigma_t}{1 + \sqrt{500\,\varepsilon_i}} \qquad i = 1, 2, 3 \tag{24}
$$

where σ_t is the concrete mean tensile strength [7]. It is assumed that aggregate interlock can carry any shear stress in the crack. It is assumed that the concrete principal stresses and the principal strains have the same direction.

The principal strains ε_1, ε_2 and ε_3 are the eigenvalues of the strain tensor.

$$
\begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = P \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} P^{-1}. \tag{25}
$$

From Eqs (22) to (25) the strain tensor can be solved numerically by the Newton-Raphson method.

The Model Code 90 is applied for computing crack widths [8]. The mean crack spacings s for uniaxial tension in the reinforcement directions are

$$s_x = \frac{2}{3}\frac{d_x}{3.6\,\rho_x} \qquad s_y = \frac{2}{3}\frac{d_y}{3.6\,\rho_y} \qquad s_z = \frac{2}{3}\frac{d_z}{3.6\,\rho_z}, \tag{26}$$

where d_x, d_y, d_z are the diameters of the reinforcing bars in the x, y, z direction. The crack spacing s in principal direction i is computed from

$$\frac{1}{s_i} = \frac{|\cos\alpha_i|}{s_x} + \frac{|\cos\beta_i|}{s_y} + \frac{|\cos\gamma_i|}{s_z} \qquad i = 1, 2, 3. \tag{27}$$

The mean crack width in the principal direction i is

$$w_i = s_i(\varepsilon_i - \varepsilon_c - \varepsilon_s) \qquad i = 1, 2, 3 \tag{28}$$

where ε_c is the concrete strain and ε_s is the concrete shrinkage. The value of ε_c is positive and the value of ε_s is negative. For simplicity, in this paper is assumed that they cancel each other out. The crack width is limited to a maximum value.

$$w_i \le w_{\max} \qquad i = 1, 2, 3. \tag{29}$$

which puts a constraint on the reinforcement ratios ρ_x, ρ_y, ρ_z. It is noted that the formulation is suitable for any consistent set of units, for example newtons and millimeters or pounds and inches. A numerical implementation for computing the crack width is shown in Appendix 3. The optimisation problem is too complicated for analytical solution.

8 Overview

The complete optimisation problem for reinforcement design is summarised in this section.

Minimise the total reinforcement ratio $\rho_x + \rho_y + \rho_z$ fulfilling six constraints.

The constraints are [2]

$\rho_x \geq 0, \quad \rho_y \geq 0, \quad \rho_z \geq 0$,

$\sigma_{c1} \leq 0$	for all load combinations related to the ultimate limit state,
$\dfrac{\sigma_{c3}}{\sigma_c} + \dfrac{\sigma_{c1}}{\sigma_t} \leq 1$	for all load combinations related to the ultimate limit state,
$w \leq w_{\max}$	for all load combinations related to the serviceability limit state.

The largest concrete principal stress σ_{c1} and the smallest concrete principal stress σ_{c3} are a function of the stress state σ_{xx} , σ_{yy} , σ_{zz} , σ_{xy} σ_{xz} , σ_{yz} , of the reinforcement ratios ρ_x , ρ_y , ρ_z and of the yield stress of the reinforcing bars σ_y , which can be larger or smaller than zero.

The crack width w is a function of the stress state σ_{xx} , σ_{yy} , σ_{zz} , σ_{xy} σ_{xz} , σ_{yz} , of the reinforcement ratios ρ_x , ρ_y , ρ_z , of Young's moduli of steel E_s and concrete E_c , of the tensile strength of concrete σ_t and of the reinforcing bar diameters d_x , d_y , d_z . The stress states differ for each load combination.

9 Examples

Table 1 shows results of the proposed optimisation problem. The rows contain computation examples. The reinforcement yield stress is σ_y = 500 N/mm² for each example. The concrete tensile strength is σ_t = 3 N/mm². The concrete uniaxial compressive strength is σ_c = – 40 N/mm². The maximum mean crack width is $w_{\max}$ = 0.2 mm. The bar diameters are $d_x = d_y = d_z$ = 16 mm. Young's moduli of steel and compressed concrete are E_s = 210000 N/mm² and E_c = 30000 N/mm².

---

[2] In the first three constraints a minimum reinforcement ratio can be included.

Column 1 contains the example numbers. Columns 2 to 7 contain the input stress states. All stresses in the table have the unit N/mm². Column 8 shows whether a stress state belongs to the ultimate or serviceability limit state. Column 9 to 11 contain the linear elastic principal stresses. Column 13 to 15 contain the output reinforcement ratios in %. Column 16 to 18 contain the principal concrete stresses. Column 19 shows the numbers of the equations in Section 4 that give the same result. It is noted that sometimes different equations in Section 4 produce the same optimal result. Column 20 shows which load combinations influence the computed reinforcement ratios.

Example 1 and 2 have also been studied by Foster et al. [4]. In example 1 the same results have been found. In example 2, Foster found ρ_x = 0.75%, ρ_y = 0, ρ_z = 0.75%. Table 2 shows that the optimal reinforcement differs considerably. However, the total reinforcement is almost the same (Foster; 0.75 + 0.00 + 0.75 = 1.50%, Table 2; 0.89 + 0.00 + 0.57 = 1.46%). Example 3 to 5 show that edge solutions and corner solutions can provide the optimal reinforcement. Comparison of example 6 and 7 shows that double stress requires twice the amount of reinforcement. Apparently, the amount of reinforcement is linear in the load factor; Example 8 and 9 show that interior solutions can provide the optimal reinforcement solution.

Example 12 consists of two load combinations. The volume reinforcement ratio is $\rho_x + \rho_y + \rho_z$ = 3.00 + 0.33 + 0.00 = 3.33%. Alternatively, we could have selected the envelope of the reinforcement requirements for the individual load combinations, which are examples 10 and 11. The volume reinforcement ratio applying the envelope method is max(3.00, 1.00) + max(0.00, 1.00) = 4.00%. Consequently, the envelope method gives a safe approximation but it overestimates the required reinforcement substantially.

Example 13 shows an uniaxial compressive force that is larger than the concrete compressive strength. The algorithm computes that the minimum reinforcement solution is 0.75% confinement reinforcement in both lateral directions. For compression reinforcement would be needed (90 – 40)/500 = 10.00% which is much larger than 0.75 + 0.75 = 1.50%. Example 14 shows that for large isotropic compression no reinforcement is needed. Example 15 considers the double amount of elastic stress of example 7. It shows that the required reinforcement is more than double because confinement reinforcement is needed. This high reinforcement ratio can be required in columns.

Table 1. Computation examples

	σ_{xx}	σ_{yy}	σ_{zz}	σ_{xy}	σ_{xz}	σ_{yz}		σ_1	σ_2	σ_3
1	2	3	4	5	6	7	8	9	10	11
1	2	-2	5	6	-4	2	ULS	8.28	4.32	-7.60
2	-3	-7	.	6	-4	2	ULS	3.28	-0.68	-12.60
3	-1	-7	10	.	.	5	ULS	11.36	-1.00	-8.36
4	3	.	10	.	5	.	ULS	12.60	0.40	.
5	10	7	-3	3	1	-2	ULS	11.86	5.71	-3.57
6	4	-7	3	7	.	-5	ULS	8.48	3.31	-11.79
7	8	-14	6	14	.	-10	ULS	16.97	6.62	-23.59
8	1	.	3	10	-8	7	ULS	10.90	8.66	-15.56
9	.	.	.	10	8	7	ULS	16.37	-6.62	-10.11
10	15	.	.	.	.	.	ULS	15.00	.	.
11	.	.	.	5	.	.	ULS	5.00	.	-5.00
12	15	.	.	.	.	.	ULS	15.00	.	.
	.	.	.	5	.	.	ULS	5.00	.	-5.00
13	-90	.	.	.	.	.	ULS	.	.	-90.00
14	-90	-90	-90	.	.	.	ULS	-90.00	-90.00	-90.00
15	16	-28	12	28	.	-20	ULS	33.94	13.23	-47.17
16	10	7	-3	3	1	-2	SLS	11.86	5.71	-3.57
17	2	-2	5	6	-4	2	ULS	8.28	4.32	-7.60
	-2	1	3	.	3	5	ULS	7.68	-0.97	-4.71
	2	1	3	4	2	.	ULS	6.26	2.58	-2.85
	1	-1	3	3	-2	1	SLS	4.39	2.36	-3.76
	-1	1	2	.	2	3	SLS	4.95	-0.22	-2.73

The dots (.) represent zeros (0) in order to improve readability of the table.

Table 2. Strains of the SLS examples

	ε_{xx}	ε_{yy}	ε_{zz}	γ_{xy}	γ_{xz}	γ_{yz}
16	0.001572	0.001357	-0.000033	0.003243	-0.000592	-0.000754
17, 4	0.000939	0.000278	0.000707	0.001387	-0.001827	-0.000934
17, 5	0.000294	0.000710	0.000956	0.001056	0.001351	0.001992

12	ρ_x 13	ρ_y 14	ρ_z 15	σ_{c1} 16	σ_{c2} 17	σ_{c3} 18	Eq. 19	decisive 20
1	2.40	0.40	1.40	.	-0.79	-15.21	14	yes
2	0.89	.	0.57	.	-2.53	-14.77	10+	yes
3	.	.	2.71	.	-1.00	-10.57	5	yes
4	1.60	.	3.00	.	.	-10.00	10–, 13, 16	yes
5	2.53	2.13	.	.	-2.02	-7.31	11–	yes
6	2.20	1.00	1.60	.	-5.76	-18.24	14	yes
7	4.40	2.00	3.20	.	-11.51	-36.49	14	yes
8	2.49	1.75	1.72	.	.	-25.78	17	yes
9	3.60	3.40	3.00	.	-22.35	-27.65	13	yes
10	3.00	.	.	.	.	.	7, 10– 17	yes
11	1.00	1.00	.	.	.	-10.00	11–, 13 ,14	yes
12	3.00	0.33	.	.	.	-1.67		yes
	3.00	0.33	.	.	.	-16.67		yes
13	.	0.75	0.75	-3.76	-3.76	-90		yes
14	.	.	.	-90	-90	-90		no
15	9.57	4.01	7.20	-2.58	-26.91	-74.41		yes
16	3.42	3.26	.	-2.41	-5.04	-11.94		yes
17	1.51	2.01	2.15	-0.59	-5.82	-16.94		no
	1.51	2.01	2.15	-2.59	-9.42	-14.35		no
	1.51	2.01	2.15	-2.40	-7.98	-11.97		no
	1.51	2.01	2.15	-4.43	-7.75	-13.17		yes
	1.51	2.01	2.15	-5.21	-8.70	-12.44		yes

Example 16 considers one serviceability limit state for which the reinforcement is only constrained by the crack width requirement. Example 17 considers linear elastic stress states due to five load combinations. Three of these are related to the ultimate limit state and two are related to the serviceability limit state. In this example the serviceability load combinations determine the computed reinforcement.[3]

Table 2 presents the strains of the SLS stress states in order to facilitate checking of the crack width computations.

10 Conclusions

A simple and safe formula for choosing reinforcement ratios ρ in the x, y and z direction is

$$\rho_x = \rho_y = \rho_z = \frac{\sigma_1}{\sigma_y}.$$

where, σ_1 is the largest principal stress as computed by the linear elastic finite element method and σ_y is the yield stress of the reinforcing bars.[4] An advantage of this reinforcement is that few extra cracks are formed when the load increases towards the ultimate load. However, this formula will overestimate the required reinforcement almost always considerably.

In case the structure is loaded by one load combination the optimal reinforcement can be computed as the valid best of eleven analytical solutions. Formulas for these solutions and a validity check have been derived and are presented in this paper. However, few structures are loaded by just one load combination.

For multiple load combinations the optimal reinforcement solution cannot be derived as simple closed form formulas. As a solution, it would be possible to compute the envelope of requirements of the individual load combinations. A similar envelope method is being used in many commercially available programs for designing plate reinforcement. In this paper it is shown that the envelope method used on three-dimensional reinforcement can result in a considerable overestimation of the required reinforcement.

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[3] The authors experienced that temperature stresses and imposed displacements, such as foundation settlements, need to be ignored in reinforcement design for the ultimate limit state. These load cases need to be included only in load combinations for the serviceability limit state. For these load cases it is important to accurately estimate the cracked stiffness that is used in the linear elastic finite element analysis.

[4] For practical use, all formulas and algorithms in this paper need to be complemented with suitable partial safety factors.

A formulation is proposed for computing the optimal reinforcement for multiple load combinations. Included are compression reinforcement, confinement reinforcement and crack control for the serviceability limit state. The optimal reinforcement results of 17 stress states are presented. The results correctly show that confinement reinforcement is much more effective than compression reinforcement.

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Notations

d_x , d_y , d_z …………………………….. reinforcing bar diameter in the x, y and z direction

E_c , E_s …………………………………. Young's modulus of concrete and steel

I_1 , I_2 , I_3 …………………………….... invariants of the linear elastic stress tensor

I_{c1} , I_{c2} , I_{c3} ………………………….. invariants of the concrete stress tensor

P ………………………………………. rotation matrix

s_1 , s_2 , s_3 …………………………….... mean crack spacing in the principal directions

s_x , s_y , s_z ……………………………… mean crack spacing in the x, y and z direction

w ……………………………………… mean crack width

$w_{\max}$ ………………………………… allowable crack width

α , β, γ ………………………………… angles of a vector with the x, y and z direction

ε_1 , ε_2 , ε_3 …………………………….. principal strains

ε_{xx} , ε_{yy} , ε_{zz} , γ_{xy} , γ_{xz} , γ_{yz} ………….. average strains

ε_c , ε_s …………………………………. concrete strain and concrete shrinkage

ρ_x , ρ_y , ρ_z ……………………………. reinforcement ratios in x, y and z direction

σ_1 , σ_2 , σ_3 …………………………… linear elastic principal stresses

σ_c ……………………………………… concrete compressive strength (negative value)

σ_{c1} , σ_{c2} , σ_{c3} ………………………… concrete principal stresses

σ_{cx} , σ_{cy} , σ_{cz} …………………………. concrete normal stresses

σ_{sx} , σ_{sy} , σ_{sz} …………………………..… reinforcing steel normal stresses

σ_t ……………………………………… concrete tensile strength

σ_{xx} , σ_{yy} , σ_{zz} , σ_{xy} , σ_{xz} , σ_{yz} ……….. linear elastic stresses

σ_y …………………………………….... steel yield stress

Appendix 1. Stress theory

The stress in a material point can be represented by a stress tensor.

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \tag{30}$$

The principal values of a stress state are the eigenvalues of the stress tensor. In this paper they are ordered, σ_1 being the largest principal stress.

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \tag{31}$$

The invariants of the stress tensor are

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2 \\ I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{xz}\sigma_{yz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2 \end{aligned} \tag{32}$$

In fact, I_3 is the determinant of the stress tensor. The invariants can be expressed in the principal stresses.

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{33}$$

The principal stresses and the invariants have the property that they are independent of the selected reference system x, y, z.

Appendix 2. Source code ULS

This appendix contains the Pascal source code for computing whether constraint 4 and 5 in Section 8 are fulfilled. The program uses a procedure "Jacobi" that computes eigen values and eigen vectors applying the Jacobi algorithm.

```
function CheckULS(sxx,syy,szz,sxy,sxz,syz,rx,ry,rz,sy,sc,st: double): boolean;
function PS(rx,ry,rz: double): boolean;
var
  a,v:          matrix;  // stress tensor, matrix with principal direction vectors
  sc1,sc2,sc3,           // concrete principal stresses
  t:            double;
begin
  a[1,1]:=sxx-rx*sy; a[1,2]:=sxy;       a[1,3]:=sxz;
  a[2,1]:=sxy;       a[2,2]:=syy-ry*sy; a[2,3]:=syz;
  a[3,1]:=sxz;       a[3,2]:=syz;       a[3,3]:=szz-rz*sy;
  Jacobi(a,v, 0.001);
  sc1:=a[1,1]; sc2:=a[2,2]; sc3:=a[3,3];
  if sc3>sc1 then begin t:=sc3; sc3:=sc1; sc1:=t end;
  if sc3>sc2 then begin t:=sc3; sc3:=sc2; sc2:=t end;
  if sc2>sc1 then begin t:=sc2; sc2:=sc1; sc1:=t end;
  if (sc1<0) and (sc3/sc+sc1/st<1) then PS:=true else PS:=false
end; // of PS
begin
  CheckULS:=false;
  if PS( rx, ry, rz) then CheckULS:=true;
  if PS(-rx, ry, rz) then CheckULS:=true;
  if PS( rx,-ry, rz) then CheckULS:=true;
  if PS( rx, ry,-rz) then CheckULS:=true;
  if PS(-rx,-ry, rz) then CheckULS:=true;
  if PS(-rx, ry,-rz) then CheckULS:=true;
  if PS( rx,-ry,-rz) then CheckULS:=true;
  if PS(-rx,-ry,-rz) then CheckULS:=true
end; // of CheckULS
```

Appendix 3. Source code SLS

This appendix contains the Pascal source code for computing whether constraint 6 in Section 8 is fulfilled. The program uses a procedure "Jacobi" that computes eigen values and eigen vectors applying the Jacobi algorithm.

```
function CheckSLS(sxx,syy,szz,sxy,sxz,syz,rx,ry,rz,st,Es,Ec,dx,dy,dz,wmax: double): boolean;
var
  i:         integer;
  a,                              // strain tensor
  v:         matrix;              // matrix with principal direction vectors
  e1,e2,e3,                       // concrete principal strains
  a1,a2,a3, b1,b2,b3, c1,c2,c3,   // concrete principal directions
  ecr,                            // concrete cracking strain
  sc1,sc2,sc3,                    // concrete principal stresses
  ssx,ssy,ssz,                    // steel stresses
  exx,eyy,ezz,gxy,gxz,gyz,        // strains
  sxxt,syyt,szzt,sxyt,sxzt,syzt,  // temporary stresses
  dsxx,dsyy,dszz,dsxy,dsxz,dsyz,  // residual stresses
  d,                              // residual stress error
  h,                              // largest possible crack spacing
  sx,sy,sz,                       // crack spacings in the x, y and z direcion
  s1,s2,s3,                       // crack spacings in the principal directions
  w1,w2,w3,                       // crack widths in the principal directions
  w:          double;             // largest crack width
begin
  // Concrete strains
  exx:=sxx/Ec;
  eyy:=syy/Ec;
  ezz:=szz/Ec;
  gxy:=sxy/Ec*2.0;
  gxz:=sxz/Ec*2.0;
  gyz:=syz/Ec*2.0;
  i:=0;
  repeat
    i:=i+1;
    // concrete principal strains and directions
    a[1,1]:=exx;   a[1,2]:=gxy/2; a[1,3]:=gxz/2;
    a[2,1]:=gxy/2; a[2,2]:=eyy;   a[2,3]:=gyz/2;
    a[3,1]:=gxz/2; a[3,2]:=gyz/2; a[3,3]:=ezz;
    Jacobi(a,v,0.000001);
    e1:=a[1,1]; e2:=a[2,2]; e3:=a[3,3];
    a1:=v[1,1]; a2:=v[1,2]; a3:=v[1,3];
    b1:=v[2,1]; b2:=v[2,2]; b3:=v[2,3];
    c1:=v[3,1]; c2:=v[3,2]; c3:=v[3,3];
    // material stresses
    ecr:=st/Ec;
    if e1<ecr then sc1:=Ec*e1 else sc1:=st/(1+sqrt(500*e1));
    if e2<ecr then sc2:=Ec*e2 else sc2:=st/(1+sqrt(500*e2));
    if e3<ecr then sc3:=Ec*e3 else sc3:=st/(1+sqrt(500*e3));
```

```
    ssx:=Es*exx;
    ssy:=Es*eyy;
    ssz:=Es*ezz;
    // total stresses
    sxxt:=a1*a1*sc1 +a2*a2*sc2 +a3*a3*sc3 +ssx*rx;
    syyt:=b1*b1*sc1 +b2*b2*sc2 +b3*b3*sc3 +ssy*ry;
    szzt:=c1*c1*sc1 +c2*c2*sc2 +c3*c3*sc3 +ssz*rz;
    sxyt:=a1*b1*sc1 +a2*b2*sc2 +a3*b3*sc3;
    sxzt:=a1*c1*sc1 +a2*c2*sc2 +a3*c3*sc3;
    syzt:=b1*c1*sc1 +b2*c2*sc2 +b3*c3*sc3;
    dsxx:=sxx-sxxt;
    dsyy:=syy-syyt;
    dszz:=szz-szzt;
    dsxy:=sxy-sxyt;
    dsxz:=sxz-sxzt;
    dsyz:=syz-syzt;
    d:=abs(dsxx) +abs(dsyy) + abs(dszz) +abs(dsxy) +abs(dsxz) +abs(dsyz);
    exx:=exx+ dsxx/Ec;
    eyy:=eyy+ dsyy/Ec;
    ezz:=ezz+ dszz/Ec;
    gxy:=gxy+ dsxy/Ec*2.0;
    gxz:=gxz+ dsxz/Ec*2.0;
    gyz:=gyz+ dsyz/Ec*2.0;
  until d<0.01;
  // Crack width
  h:=5000;
  if rx>0.00001 then sx:=0.1852*dx/rx else sx:=h; if sx>h then sx:=h; if sx<1.0 then sx:=1.0;
  if ry>0.00001 then sy:=0.1852*dy/ry else sy:=h; if sy>h then sy:=h; if sy<1.0 then sy:=1.0;
  if rz>0.00001 then sz:=0.1852*dz/rz else sz:=h; if sz>h then sz:=h; if sz<1.0 then sz:=1.0;
  s1:=1.0/( abs(a1)/sx +abs(b1)/sy +abs(c1)/sz );
  s2:=1.0/( abs(a2)/sx +abs(b2)/sy +abs(c2)/sz );
  s3:=1.0/( abs(a3)/sx +abs(b3)/sy +abs(c3)/sz );
  w1:=s1*e1;
  w2:=s2*e2;
  w3:=s3*e3;
  w:=0; if w1>w then w:=w1; if w2>w then w:=w2; if w3>w then w:=w3;
  if w<wmax then CheckSLS:=true else CheckSLS:=false
end; // of CheckSLS
```