Buckling of laminated glass columns

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The buckling force of a laminated glass column is highly dependent on the shear stiffness of the soft foil which connects the two glass layers. The value of the buckling force is bounded by the lower limit when no foil is present (two separate glass planes) and the upper limit when the foil is infinitely stiff (the glass layers are ideally coupled). A formula for the buckling force should match these ultimate values. Three formulas for the buckling force of a laminated glass column exist. One of them produces the correct lower and upper limit, one produces the correct lower limit and a conservative upper limit and one produces a zero lower limit and conservative upper limit. The way they appear in the literature, the formulas do not provide engineering insight and do not contain a distinct parameter which controls the transition from the lower to the upper limit. In the present article, an alternative formula is derived on the basis of a set of three simultaneous differential equations, supposition of a sine/cosine displacement field, and the formulation of an eigenvalue problem. The new formula is simple and provides engineering insight. Its derivation yields a dimensionless parameter which controls the transition from the lower to the upper limit. The similarities and/or differences with the existing formulas are discussed. Subsequently, initial deformation of the glass layers is taken into account, transferring the stability problem into a strength problem, so that failure is governed by tensile strength. This allows straightforward computation of the stresses and a unity check to be done.

Key words: Laminated glass, buckling formulas, derivation

1 Introduction

Laminated glass elements consist of two glass layers with a thin foil in between. As such, laminated glass is a special application of a sandwich structure. In sandwich elements often the faces are thin if compared to the core. In laminated glass the opposite occurs, the glass layers are relatively thick and the foil in between is thin. It means that one can only use the sandwich theory for thick faces. The present paper deals with one-dimensional column buckling of laminated glass panels and restricts to theoretical work. For interesting experimental work it is referred to recent publications like (Luible, 2004) and (Luible et al., 2005). When a formula for column buckling is required, researchers refer to (hand)books for sandwich construction and select the relevant formula for thick faces. Examples are found in (Allen, 1969), (Sattler et al.,

1974) and (Zenkert, 1997). The Sattler formula is also included in (Stamm et al., 1974). It is a check on the validity of the formula that the formula must produce correct values for two bounding cases. A lower limit P_L occurs when no foil is put in at all. Now the two glass planes behave independently of each other and the buckling force is simply the sum of the two separate buckling forces, so it is governed by the sum of the bending stiffness of the two individual glass planes. An upper limit P_U occurs when the foil has extremely large shear stiffness. Now the two glass planes are ideally coupled and the bending stiffness of the composed cross-section governs the buckling force. One of the three mentioned existing formulas correctly meets the requirement to be bounded by these two limit values, one does it approximately and one fails to do so. But even the formula which correctly meets the requirement, leaves us troubled with the fact that no clear dimensionless parameter controls the transition from the one limit to the other. Therefore, an alternative derivation is made, starting from the basics, with two results: (i) a charming new formula is obtained and (ii) the derivation nicely reveals the dimensionless parameter which controls the transition. The new formula is considered to be an improvement over the old ones from the viewpoint of elegance and insight.

Figure 1 shows the composition of the panel and its position with respect to the x- and z-axis. The glass material is fully linear elastic with modulus of elasticity E. The two layers, which can be different, are numbered 1 and 2. These numbers will be used as subscripts to quantities related to the glass layer cross-sections. The thicknesses are t_1 and t_2 , the areas A_1 and A_2 and the second-order moments I_1 and I_2 respectively. The foil material is thin with regard to the glass layers and is soft, so just shear stresses are considered and no normal stresses are taken into account. The shear modulus is G_s and the foil thickness t_s . Though in fact the foil is a visco-elastic material with a G_s depending on temperature and load duration, it is treated here as a linear-elastic material, which implies that the approach is valid for short load duration, like wind and impact. The glass panel has a length l and width b and is simply supported at both ends. The buckling load P_{cr} is applied in the neutral plane of the two glass layers, which occurs if they are ideally connected. This plane is on a distance e_1 to the centre line of glass layer 1 and distance e_2 to the centre line of glass layer 2. These distances are:

$$e_1 = \frac{A_2}{A} e \; ; \quad e_2 = \frac{A_1}{A} e \tag{1}$$

where

$$e = e_1 + e_2 \; ; \quad A = A_1 + A_2 \tag{2}$$

In reality no glass panel will be perfectly straight, so hereafter an initial sine-shaped imperfection will be assumed with a mid-span value w_0. When the buckling load is not applied in the neutral plane, but eccentrically, the eccentricity is simply added to the initial displacement w_0, which is not totally correct but a sufficiently close approximation in view of the fact that the value of w_0 anyhow has to be chosen on basis of good judgment.

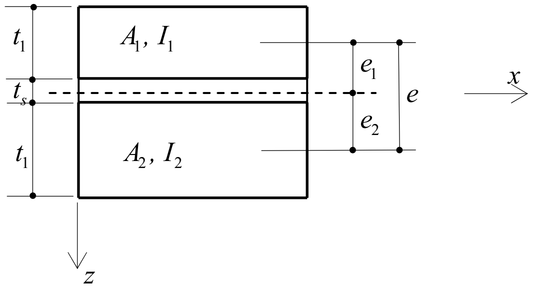

Figure 1: Geometric properties of a laminated glass element of two layers

Hereafter, in Section 2 the three existing formulas are discussed and in Section 3 the procedure for classical Euler buckling is called in mind very briefly as a basis for the derivation of the new formula in Section 4. In Section 5 the new formula is compared to the three existing ones, Section 6 discusses how an initial imperfection changes the buckling problem into a strength problem and in Section 7 an example is discussed. Conclusions are drawn in Section 8.

2 Discussion of existing formula

In this section the buckling formulas of Sattler, Zenkert and Allen are reproduced and it will be checked if they correctly match the lower and upper limit. For this purpose all three formulas are written in the same notation, and it also is convenient to introduce a few auxiliary quantities:

$D_L = EI$:	summed bending stiffness of the two individual glass layers,
$D_O = EI_O$:	bending stiffness of the joint two layers, ideally connected, excluding bending stiffness of the individual layers,
$D_U = EI_U$:	bending stiffness of the joint two layers, ideally connected, including bending stiffness of the individual layers,
D_S :	shear stiffness of the foil.

Herein:

$$I = I_1 + I_2\,; \quad I_O = \frac{e^2}{A_1^{-1} + A_2^{-1}}\,; \quad I_U = I_L + I_O\,; \quad D_S = \frac{G_s\, b\, e^2}{t_s} \tag{3}$$

Correspondingly we define buckling forces:

$$P_L = \pi^2 \frac{EI}{l^2}\,; \quad P_O = \pi^2 \frac{EI_O}{l^2}\,; \quad P_U = \pi^2 \frac{EI_U}{l^2}\,; \quad P_S = D_S \tag{4}$$

where P_L, P_O, P_U are buckling forces when shear deformation is neglected and P_S applies if only shear deformations occur. P_L is the lower limit of the buckling force, P_U the upper limit and P_O an approximation of the upper limit (namely P_U minus P_L).

Formulas in literature

The buckling formulas as copied from literature, in notation of this article, read:

Sattler et al., 1974:

$$P_{cr} = \frac{\pi^2(1 + \alpha + \pi^2 \alpha\beta)}{1 + \pi^2 \beta} \frac{D_O}{l^2} \tag{5}$$

where

$$\alpha = \frac{D_1 + D_2}{D_O}\,; \quad \beta = \frac{t_s}{G_s\, b\, e^2} \frac{D_O}{l^2} \tag{6}$$

Zenkert, 1997:

$$P_{cr} = \frac{\dfrac{\pi^4 D_L D_O}{D_S\, l^4} + \dfrac{\pi^2 D_O}{l^2}}{1 + \dfrac{\pi^2 D_O}{D_S\, l^2}} \tag{7}$$

Allen, 1969:

$$P_{cr} = \frac{P_O}{1 + \dfrac{2 P_O}{G_s\, b (t_s + t)^2 / t_s}} \tag{8}$$

Of these formulas, the ones in (5) and (7) apply generally for non-symmetric sandwich panels, while the third in (8) applies for symmetric panels only. In the form the formulas appear here, it is not easy to recognize which terms represent the contribution of the individual glass layers and which the joint action of foil and glass layers, even less whether the lower and upper limit are matched correctly. Also it is hard to compare them. In order to facilitate that, the formulas can be rewritten by appropriate manipulations into:

Sattler et al., 1974:

$$P_{cr} = \frac{P_S P_U + P_L P_O}{P_S + P_O} \tag{9}$$

Zenkert, 1997:

$$P_{cr} = \frac{P_S P_O + P_L P_O}{P_S + P_O} \tag{10}$$

Allen:

$$P_{cr} = \frac{P_S P_O}{P_S + P_O} \tag{11}$$

Now it can be checked whether the lower and upper limits are matched. The three lower limits, occurring for $D_S = 0$ (so $P_S = 0$), are P_L, P_L and 0, respectively. We conclude that the formulas of Sattler et al. and Zenkert yield the expected lower limit, whereas the formula of Allen predicts the wrong result. It predicts that the glass layers can not carry any axial load if no foil is put in. The three upper limits, occurring for $D_S \to \infty$ (so $P_S \to \infty$), are P_U, P_O and P_O, respectively.

Only the first one (Sattler et al.) is correct. The other two formulas yield an approximation for the upper value of the buckling force, because the contribution of the individual glass layers to the bending stiffness appears to be neglected in the formula for the upper limit. Whereas the difference for a sandwich construction with metal faces may be negligible, it is not for laminated glass. Then the difference can be in the order of 20%. For the rest, (i) the approximation is on the safe side, because the formulas underestimate the buckling force, and (ii) the real buckling force is in between the lower and upper limit, so the difference anyhow is smaller then 20%.

As for the background of the formulas, Sattler et al. publish the full derivation of their formula. The Allen formula is obtained by the well-known approximation that P_{cr} can be computed from the equation:

$$\frac{1}{P_{cr}} = \frac{1}{P_E} + \frac{1}{P_S} \tag{12}$$

where P_E is the Euler buckling force for pure bending. Why Allen substitutes P_O for P_E and not P_U is not clear. Zenkert also starts from formula (12), but after that accounts for the thickness of the faces by giving the formula a slightly different form, 'since it must be derived from another governing equation', without showing that equation.

3 Procedure for classical Euler buckling

In Section 4 use will be made of knowledge which is borrowed from the procedure for classical Euler buckling for a single glass layer. Therefore, this material will be discussed here briefly. We consider a simply supported ideally straight beam along the x-axis with bending stiffness EI, length l, applied force P in axial direction, distributed lateral load f and displacement w in z-direction transverse to the beam axis, see Figure 2. Only flexural deformation of the beam is taken into account. The origin of the set of axes is in the left-hand support. The differential equation for this beam is:

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = f \tag{13}$$

The first term in the left-hand part of the equation is the well-known first-order Euler beam contribution and the second term represents the second-order effect if the equilibrium is considered in the deformed state. In the derivation for the laminated glass panel hereafter the first term will be adapted, but the second term can be borrowed unchanged.

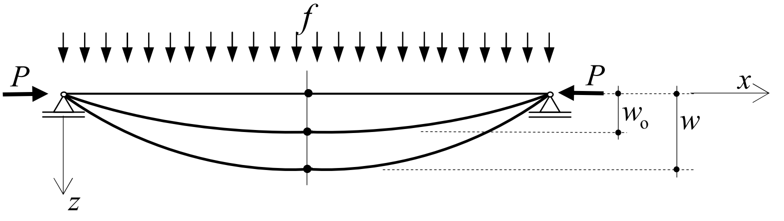

Figure 2: Initial deflection w_o and final deflection w after application of load P in a single glass element

For zero load f, a sine-shaped deflection function

$$w(x) = w\,sin(\pi x / l) \tag{14}$$

satisfies the boundary conditions $w = 0$ and $M = 0$ for displacement and bending moment respectively, and transforms differential equation (13) into:

$$\frac{\pi^2}{l^2}\left(\frac{\pi^2 EI}{l^2} - P\right) w = 0 \tag{15}$$

Buckling means that w can be nonzero in absence of loading. In case of nonzero w the left-hand member of equation (15) can vanish only if the term between brackets is zero, yielding the classical Euler solution for the critical load:

$$P_{cr} = \frac{\pi^2 EI}{l^2} \tag{16}$$

In case of a sine-shaped initial deflection with maximum value w_o the differential equation changes for zero f into:

$$EI\,\frac{d^4(w - w_o)}{dx^4} + P\,\frac{d^2 w}{dx^2} = 0 \tag{17}$$

No stability problem occurs anymore and the defection w can be solved directly. Again the solution $w(x)$ is sine-shaped and now the value w is solved from

$$\left(\frac{\pi^2 EI}{l^2} - P\right) w = \frac{\pi^2 EI}{l^2}\, w_o \tag{18}$$

Accounting for equation (16) we can rewrite (18) into:

$$(P_{cr} - P)\, w = P_{cr} w_o \tag{19}$$

If now the dimensionless parameter n is introduced according to:

$$n = \frac{P_{cr}}{P} \tag{20}$$

(always larger than 1), we solve from equation (19):

$$w = \frac{n}{n-1}\, w_o \tag{21}$$

The term $n/(n - 1)$ is an amplification factor for the deflection. In words, the initial deflection w_0 has grown to w due to the application of the axial compressive load P, and the stability problem has changed into a strength problem. In the next section this procedure also applies for the laminated glass case. A formula different from (16) will be found for P_{cr} but equations (19), (20) and (21) still hold true.

4 Procedure for laminated glass element

The two glass layers have been drawn in Figure 3. The foil between the layers is marked with distributed strips perpendicular to the layers. We define displacements u_1 and u_2 in the mid-plane of the respective layers and displacement w in the transverse direction, which is common to both glass layers. The displacements u_1 , u_2 and w are independent degrees of freedom. This implies that distributed loading can be applied in each of these three directions. We indeed introduce a distributed transverse load f per unit length of the glass panel in w-direction and f_1 and f_2 in the directions of u_1 and u_2 . The latter two do not occur in reality but are applied for

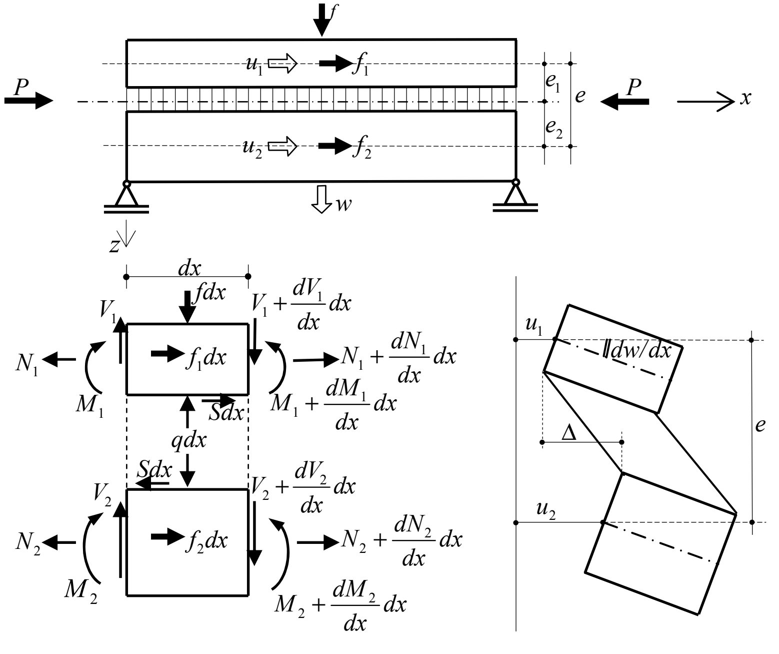

Figure 3: Definition of degrees of freedom (top), quantities playing a role in equilibrium (bottom left) and quantities playing a role in kinematics (bottom right).

the time being for reasons of completeness. In course of the derivation we will put them to zero. The load f in w-direction is applied on layer 1, but will be partly transferred to layer 2 by a distributed compression force q in the foil lateral to the glass layers.

In the layers normal forces N_1 and N_2, bending moments M_1 and M_2, and shear forces V_1 and V_2 will occur. The strains ε_1 and ε_2 due to the normal forces and the curvature κ due to the bending moments (the same curvature in both layers) are taken into account, but the deformation due to the shear forces is neglected. A shear force S per unit length occurs in the mid-plane of the foil. The deformation which is associated with S is a slip Δ between the inner faces of layer 1 and layer 2.

The scheme in Figure 4 summarizes the quantities which play a role in the laminated glass problem. It shows vectors for displacements, deformations, internal section forces and external loading respectively. The scheme is of great help in deriving the needed set of relations: (i) the kinematical relationship between displacements and deformation, (ii) the constitutive relationship between deformations and section forces, and (iii) the equilibrium relationships between the internal section forces and external loading.

$$\begin{pmatrix} u_1 \\ u_2 \\ w \end{pmatrix} \qquad \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \kappa \\ \Delta \end{pmatrix} \qquad \begin{pmatrix} N_1 \\ N_2 \\ M \\ S \end{pmatrix} \qquad \begin{pmatrix} f_1 \\ f_2 \\ f \end{pmatrix}$$

$$\underbrace{\qquad\qquad}_{\substack{\textit{kinematic}\\ \textit{relations}}} \quad \underbrace{\qquad\qquad}_{\substack{\textit{constitutive}\\ \textit{relations}}} \quad \underbrace{\qquad\qquad}_{\substack{\textit{equilibrium}\\ \textit{relations}}}$$

Figure 4: Schematic overview of quantities and relationships that govern the buckling behaviour of a laminated glass panel

The left-hand bottom part of Figure 3 shows which forces and moments act on differential glass parts of length dx. In this part of the figure the foil thickness is exaggerated. The picture shows the sign convention for all quantities. The right-hand bottom part shows in which way the slip Δ is related to the displacements u_1, u_2 and dw/dx. Because the deflection w is common to the two glass layers, it is convenient to sum up hereafter their flexural stiffness, bending moment and shear force, introducing total values EI, M and V:

$$EI = EI_1 + EI_2 \,; \quad M = M_1 + M_2 \,; \quad V = V_1 + V_2 \tag{22}$$

*4.1 **Set of three relationships***

The kinematical relationships are:

$$\varepsilon_1 = \frac{du_1}{dx}\ ; \quad \varepsilon_2 = \frac{du_2}{dx}; \quad \kappa = -\frac{d^2w}{dx^2}\ ; \quad \Delta = -u_1 + u_2 + e\frac{dw}{dx} \tag{23}$$

The constitutive relationships are:

$$N_1 = EA_1\varepsilon_1\ ; \quad N_2 = EA_2\varepsilon_2\ ; \quad M = EI\kappa\ ; \quad S = k_s\Delta \tag{24}$$

Herein the foil stiffness k_s is defined by:

$$k_s = \frac{G_s b}{t_s} \tag{25}$$

Where b and t_s are defined at page 148 and 149. The equations of equilibrium in x-direction of the two glass parts can be directly written:

$$-\frac{dN_1}{dx} - S = f_1\ ; \quad -\frac{dN_2}{dx} + S = f_2 \tag{26}$$

The relationship in z-direction between the moment M, shear force S and load f requires a number of intermediate steps. Equilibrium in rotational direction of the two glass parts requires:

$$\frac{dM_1}{dx} + e_1 S - V_1 = 0\ ; \quad \frac{dM_2}{dx} + e_2 S - V_2 = 0 \tag{27}$$

Force equilibrium in z-direction is satisfied by:

$$f - q + \frac{dV_1}{dx} = 0\ ; \quad q + \frac{dV_2}{dx} = 0 \tag{28}$$

Summing up the two relationships of (27) and similarly of (28), accounting for (22), yields:

$$\frac{dM}{dx} + eS - V = 0\ ; \quad f + \frac{dV}{dx} = 0 \tag{29}$$

The last step is to eliminate V from (29), with the result:

$$-\frac{d^2M}{dx^2} - e\frac{dS}{dx} = f \tag{30}$$

The set relationships (23), (24), (26) and (30) are the basis for the derivation of the buckling formula.

4.2 *Determination of buckling force*

Substitution of (17) into (18) makes the section forces dependent of the displacements. Subsequently this result is substituted into the equilibrium equations (26) and (30), yielding three simultaneous differential equations:

$$
\begin{aligned}
&-EA_1 \frac{d^2 u_1}{dx^2} - k_s\left(-u_1 + u_2 + e\frac{dw}{dx}\right) = f_1 \\
&-EA_2 \frac{d^2 u_2}{dx^2} + k_s\left(-u_1 + u_2 + e\frac{dw}{dx}\right) = f_2 \\
&EI \frac{d^4 w}{dx^4} - e k_s\left(-\frac{du_1}{dx} + \frac{du_2}{dx} + e\frac{d^2 w}{dx^2}\right) + P\frac{d^2 w}{dx^2} = f
\end{aligned}
\tag{31}
$$

In the third differential equation we have added the second-order term as was done earlier in equation (13). Written in matrix form for zero distributed loading f_1, f_2 and f we obtain:

$$
\begin{bmatrix}
-EA_1 \frac{d^2}{dx^2} + k_s & -k_s & -ek_s \frac{d}{dx} \\
-k_s & -EA_2 \frac{d^2}{dx^2} + k_s & ek_s \frac{d}{dx} \\
ek_s \frac{d}{dx} & -ek_s \frac{d}{dx} & EI \frac{d^4}{dx^4} + (P - e^2 k_s)\frac{d^2}{dx^2}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ w \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{32}
$$

The trial solution,

$$
u_1(x) = u_1 \cos\frac{\pi x}{l}; \quad u_2(x) = u_2 \cos\frac{\pi x}{l}; \quad w(x) = w \sin\frac{\pi x}{l}
\tag{33}
$$

satisfying the boundary conditions, transforms (32) into:

$$
\begin{bmatrix}
k_1 + k_s & -k_s & -\frac{\pi}{l} e k_s \\
-k_s & k_2 + k_s & \frac{\pi}{l} e k_s \\
-\frac{\pi}{l} e k_s & \frac{\pi}{l} e k_s & \frac{\pi^2}{l^2}\left\{P_L + (e^2 k_s - P)\right\}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ w \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{34}
$$

where P_L is defined in (4) and the 'stiffnesses' k_1 and k_2 as follows:

$$k_1 = \frac{\pi^2 E A_1}{l^2}\ ; \quad k_2 = \frac{\pi^2 E A_2}{l^2} \tag{35}$$

Buckling occurs if nonzero displacements u_1 , u_2 and w can occur for zero loading. This is the case when the determinant of the stiffness matrix in (34) becomes zero. Elaboration of this determinant delivers (after multiplication by l^2/π^2):

$$(k_1 k_2 + k_1 k_s + k_2 k_s)P_L + k_1 k_2 k_s e^2 - (k_1 k_2 + k_1 k_s + k_2 k_s)P_{cr} = 0 \tag{36}$$

From this we solve the surprisingly simple buckling formula:

$$P_{cr} = P_L + \frac{1}{f_1 + f_s + f_2} e^2 \tag{37}$$

where the flexibilities in the denominator are:

$$f_1 = k_1^{-1}\ ; \quad f_s = k_s^{-1}\ ; \quad f_2 = k_2^{-1} \tag{38}$$

It is easily understood that the first term in the right-hand member of equation (37) is due to the bending moments and the second term to the normal forces in the layers. The lower and upper limit value of this formula correspond with $k_s = 0$ and $k_s \rightarrow \infty$ respectively, so $f_s \rightarrow \infty$ and $f_s = 0$:

Lower limit: P_L

Upper Limit: $P_U = P_L + \dfrac{1}{f_1 + f_2} e^2$ (39)

It is concluded that formula (37) satisfies the condition that $P_{cr} = P_L$ for zero shear stiffness k_s of the foil (very large f_s) and $P_{cr} = P_U$ for very large shear stiffness k_s (zero f_s). The latter is easily seen from a comparison with equations (3) and (4), reminding the definition of k_1 and k_2 in (35). The structure of relationship (37) facilitates to write the buckling force P_{cr} as an interpolation between the lower and upper limit. If we introduce:

$$\xi = \frac{f_1 + f_2}{f_1 + f_s + f_2} \tag{40}$$

the equation (37) can be transformed into:

$$P_{cr} = (1-\xi)P_L + \xi P_U \tag{41}$$

The transition from P_L to P_U occurs linearly, see Figure 5, and ξ is the controlling dimensionless parameter. This parameter is the ratio between on the one hand the sum of the flexibilities of the two glass layers and on the other hand the sum of the flexibilities of both the two glass layers and the foil. If the foil stiffness k_s varies from zero to infinity, the parameter ξ runs from 0 to 1.

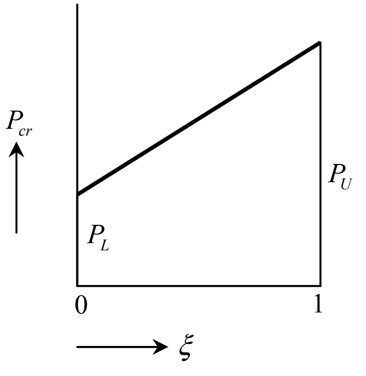

Figure 5: The dimensionless parameter ξ controls a linear transition from the lower limit buckling limit to the upper one.

5 Comparison of new formula and existing formulas

In hindsight it appears that the formula of Sattler et al. in (5) and the new one in (37) are in fact the same formula. The formula of Zenkert is equal to the new formula in the lower limit, but becomes gradually smaller than the new one for increasing foil stiffness; the upper limit is P_L smaller. The formula of Allen produces a value of the buckling force that is P_L smaller than the new one for each value of the foil stiffness. This and that is seen best if we start from the formula of Sattler et al. in (9). Substitution of $P_U = P_L + P_O$ changes it into:

$$P_{cr} = \frac{(P_S + P_O)P_L}{P_S + P_O} + \frac{P_S P_O}{P_S + P_O} = P_L + \frac{P_S P_O}{P_S + P_O} \tag{42}$$

After division of numerator and denominator of the last term by $P_S P_O$ and subsequent substitution of (4) we obtain:

$$P_{cr} = P_L + \frac{1}{\dfrac{1}{P_O} + \dfrac{1}{P_S}} = P_L + \frac{1}{\dfrac{l^2}{\pi^2 EI_O} + \dfrac{1}{D_S}} \tag{43}$$

Accounting for the relationships (3), (25), (35) and (38) the formula changes into:

$$P_{cr} = P_L + \frac{e^2}{\dfrac{l^2}{\pi^2 EA_1} + \dfrac{l^2}{\pi^2 EA_2} + \dfrac{1}{k_s}} = P_L + \frac{1}{f_1 + f_s + f_2} e^2 \tag{44}$$

Indeed, this Sattler equation is equal to the freshly derived formula in (37). As a consequence, relationship (42) also is a representation of the new formula. If compared to formula (11) of Allen, it clarifies that he misses the contribution P_L , regardless the value of the foil stiffness. Summing up, the formula of Sattler et al. is correct, but in the opinion of the author the freshly derived formula is preferable from the viewpoint of elegance and insight.

6 Unity check on stresses

The initial imperfection in real laminated glass panels makes that failure is not controlled by stability but strength. The compressive stress of glass is very high if compared to the tensile strength, hence the tensile strength governs the applicable compressive force (Luible, 2004). Given an axial load P and an initial deflection w_o mid-span, the final deflection w is calculated from equations (20) and (21):

$$n = \frac{P_{cr}}{P}\ ; \quad w = \frac{n}{n-1} w_o \tag{45}$$

where P_{cr} is defined in (37). The total moment mid-span:

$$M_{\text{total}} = P\,w \tag{46}$$

is split in two contributions, one which consists of the sum M of the bending moments and one which is due to the normal forces N in the layers :

$$M_{\text{total}} = M_M + M_N \tag{47}$$

The two parts in the right-hand member of (44) can be easily determined because their ratio is equal to the ratio of the terms in equation (37) for P_{cr} . Here we restrict ourselves to the case of a positive w_o value and consider the face of the layers with positive normal in the z-direction, here called bottom face. Tensile stresses definitely will occur at the bottom face of layer 2, but the stress in the bottom face of layer 1 should also be examined. The moment M_M divided over layer 1 and 2 proportional to their flexural stiffness:

$$M_1 = \frac{EI_1}{EI} M_M \, ; \quad M_2 = \frac{EI_2}{EI} M_M \tag{48}$$

The moment M_N yields normal forces:

$$N_1 = -N_e \, ; \quad N_2 = N_e \quad \text{where} \quad N_e = \frac{M_N}{e} \tag{49}$$

The glass layers are further loaded by a compressive force P which is split in P_1 and P_2 in the ratio A_1 to A_2 . The unity check in the bottom face of layer 1 and layer 2 is respectively:

$$\frac{-N_e - P_1}{f_t A_1} + \frac{M_1}{f_t W_1} \leq 1 \, ; \quad \frac{N_e - P_2}{f_t A_2} + \frac{M_2}{f_t W_2} \leq 1 \tag{50}$$

where W_1 and W_2 are the section moduli of the layers and f_t is the design tensile strength. Most likely the latter check in (47) is the critical one.

7 Example

We consider a simply supported glass panel 10/1.5/10 mm of length l = 1500 mm and width b = 1000 mm. The glass elasticity modulus is E = 70000 N/mm² and the foil shear modulus G_s = 0.5 N/mm². An initial deflection of $w_o = l/400$ = 1500/400 = 3.75 mm is assumed, as is recommended in (Luible, 2004). The supposed load is 50 kN and a unity check will be done. Similar to the European code design preEN 13474-2-2:2000 E for float glass, an allowable tensile stress f_t = 17 N/mm² is chosen.

e = 5 + 1.5 + 5 = 11.5 mm.

$A_1 = A_2$ = 10 . 1000 = 10000 mm².

$I_1 = I_2$ =1000 . 10 ³/12 = 83333 mm⁴.

Calculation of stiffnesses

$$k_1 = \frac{\pi^2 EA_1}{l^2} = \frac{\pi^2 \cdot 70000 \cdot 10000}{1500^2} = 3070.5 \text{ N/mm}^2.$$

$$k_2 = \frac{\pi^2 EA_2}{l^2} = \frac{\pi^2 \cdot 70000 \cdot 10000}{1500^2} = 3070.5 \text{ N/mm}^2.$$

$$k_s = \frac{G_s\, b}{t_s} = \frac{0.5 \cdot 1000}{1.5} = 333.33 \;\; \text{N/mm}^2.$$

Calculation of limits

$$P_L = \frac{\pi^2 E(I_1 + I_2)}{l^2} = \frac{\pi^2 \cdot 70000 \cdot 166667}{1500^2} = 51178 \text{ N.}$$

$$P_U = P_L + \frac{e^2}{f_1 + f_2} = 51178 + \frac{11.5^2}{3070.5^{-1} + 3070.5^{-1}} = 51178 + 203037 = 254215 \text{ N.}$$

Calculation of parameter ξ and buckling force P_{cr}

$$\xi = \frac{f_1 + f_2}{f_1 + f_s + f_2} = \frac{3070.5^{-1} + 3070.5^{-1}}{3070.5^{-1} + 333.33^{-1} + 3070.5^{-1}} = \frac{0.65136 \cdot 10^{-3}}{0.36514 \cdot 10^{-3}} = 0.17839$$

$$P_{cr} = (1 - \xi) P_L + \xi P_U = 0.82161 \cdot 51178 + 0.17839 \cdot 254215 \text{ N.}$$

$$P_{cr} = 42048 + 45349 = 87397 \;\; \text{N.}$$

Calculation of amplification factor

$$n = \frac{P_{cr}}{P} = \frac{87397}{50000} = 1.7479 \quad \rightarrow \quad \frac{n}{n-1} = \frac{1.7479}{0.7479} = 2.337$$

Calculation of stresses and unity check in layer 2.

$$w = \frac{n}{n-1} w_o = 2.337 \cdot 3.75 = 8.7638 \text{ mm.}$$

$$M = wP = 8.7638 \cdot 50000 = 438190 \text{ Nmm.}$$

$$M = M_M + M_N = \frac{51178}{87397} 438190 + \frac{36219}{87397} 438190 = 256596 + 181594 \text{ Nmm.}$$

$$N_e = \frac{M_N}{e} = \frac{181594}{11.5} = 15788 \;\; \text{N.}$$

$W = b\,t_2^2 / 6 = 1000 \cdot 10^2 / 6 = 16667$ mm³.

The stresses are calculated from the equation:

$$\sigma = \pm \frac{M_M}{W_i} + \frac{N_e - 0.5P}{A_i} = \pm \frac{256596}{16667} + \frac{25345 - 25000}{10000} = \pm 15.4 + 0.0 \ \text{N/mm}^2 \ (i = 1, 2).$$

The result of the stress calculation is shown in Figure 6. The unity check for glass layer 2 reads:

$$\frac{M_M}{f_t W} + \frac{N_e - 0.5P}{f_t A_2} = \frac{256596}{17 \cdot 16667} + \frac{25345 - 25000}{17 \cdot 10000} = 0.91 < 1$$

The unity check is satisfied, so the supposed load can be carried.

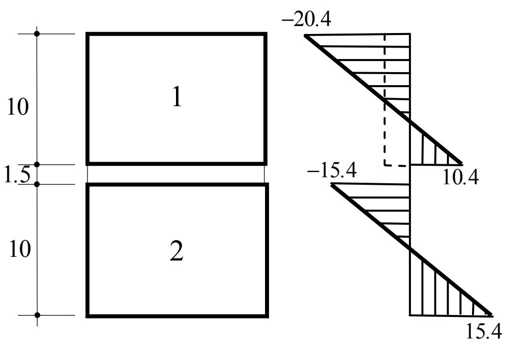

Figure 6: Stresses in the glass layers for the discussed example (length in mm and stress in N/mm²).

8 Conclusions

Of three considered formulas for the buckling force of a laminated glass column, one produces the correct lower limit (at zero foil stiffness) and upper limit (at very stiff foil), one produces the correct lower limit but a conservative upper limit (then missing the contribution of the individual glass layers), and one always produces a conservative value (missing the contribution of the individual glass layers for any foil stiffness). As they appear in the literature, the formulas do not provide engineering insight and do not show which parameter controls the transition from lower limit (uncoupled glass layers) to upper limit (ideally connected layers). The freshly derived new formula for the buckling force does it in an elegant way, is simple and provides insight. The derivation yielded the dimensionless parameter, which controls the contribution of the foil between the two glass layers. A unity check for tensile stress can be done on the basis of a proper choice of an initial deflection.

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