Stresses near a plate vertex due to a shear force on one of the edges

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A closed form solution is presented for the stresses near a rectangular vertex of linear elastic plate loaded by an evenly distributed shear force on one of the edges. The force method and the Airy stress function have been applied in the derivations. Remarkable is that the stress in the vertex depends on the direction in which the vertex is approached. The consequences for structural analysis are discussed.

Key words: Elasticity theory, Airy stress function, biharmonic equation, singular point, plain stress, plain strain, polar coordinates

1 Introduction

Many elementary situations in structural engineering have closed form solutions for the stresses assuming linear elastic material behaviour [Timoshenko 1959, Timoshenko 1971, Barber 1993]. For example the stresses in a half space due to a concentrated surface force. These solutions are used in code regulations and preliminary design of many types of structure. They have been derived before the introduction of finite element computer programs and mathematical computer programs. The historical derivations took many hours of accurate work [Meleshko 2003], however, to date they can be performed by modern mathematical software in less than a second. Therefore, computers make it possible to explore many more problems for closed form solutions. One of these problems is the subject of this paper.

Consider a rectangular vertex of an infinitely large plate (Fig. 1). One of the plate edges is loaded by an evenly distributed shear force s. The plate thickness is t. The coordinate axes are directed along the plate edges.

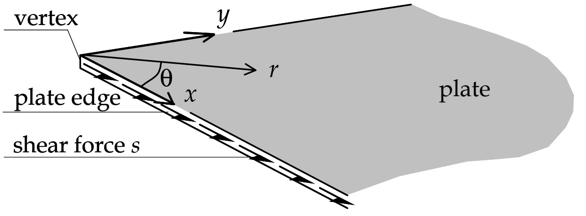

Figure 1: Shear force loading on one edge of a plate near a rectangular vertex

As shown in the following sections, the stress distribution in the plate shows a compressive stress of

$$\sigma_{xx} = -\frac{\pi s}{2t} \tag{1}$$

everywhere along the loaded edge and a tensile stress of

$$\sigma_{yy} = \frac{\pi s}{2t} \tag{2}$$

everywhere along the not loaded edge. This is also the largest principal stress σ_1 . The smallest principal stress occurs everywhere along the loaded edge.

$$\sigma_2 = -\frac{s}{t}\frac{1}{4}(\pi + \sqrt{16 + \pi^2}) = -2.06\frac{s}{t} \tag{3}$$

The largest comparison stress according to the failure criterion of Von Mises is

$$\sigma_{VM} = \frac{s}{t}\sqrt{3 + \frac{1}{4}\pi^2} = 2.34\frac{s}{t}, \tag{4}$$

which also occurs everywhere along the loaded edge.

2 The Force Method

In elasticity theory the force method is successfully applied to analyse plates loaded in plain stress or plane strain [Den Hartog 1987]. In this method we look for an Airy stress function ϕ that fulfils the biharmonic equation

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0\,. \tag{5}$$

When found, the plate forces are derived by

$$n_{xx}=\frac{\partial^2\phi}{\partial y^2},\quad n_{yy}=\frac{\partial^2\phi}{\partial x^2},\quad n_{xy}=-\frac{\partial^2\phi}{\partial x\partial y} \tag{6}$$

where $n_{xx}=t\sigma_{xx}$, $n_{yy}=t\sigma_{yy}$, $n_{xy}=t\sigma_{xy}$ and t is the plate thickness. For the situation analysed in this paper the boundary conditions are

$$\begin{aligned} x=0 &\quad\rightarrow\quad \frac{\partial^2\phi}{\partial y^2}=0,\quad \frac{\partial^2\phi}{\partial x\partial y}=0\\ y=0 &\quad\rightarrow\quad \frac{\partial^2\phi}{\partial x^2}=0,\quad \frac{\partial^2\phi}{\partial x\partial y}=s \end{aligned} \tag{7}$$

The following stress function fulfils the biharmonic equation (5) and the boundary conditions (7).

$$\phi=\tfrac{1}{2}s\left[xy+\tfrac{1}{2}\pi x^2-(x^2+y^2)\arctan\frac{x}{y}\right] \tag{8}$$

It has been obtained by trail and error, starting with an assumption for the shear stress distribution n_{xy} and deriving the normal stresses n_{xx} and n_{yy} from equilibrium and the boundary conditions.

The stress function can also be written in the cylinder coordinates r and θ using $x = r\cos\theta$ and $y = r\sin\theta$.

$$\phi = \tfrac{1}{2} s\, r^2 \left[\sin\theta\cos\theta - \tfrac{1}{2}\pi \sin^2\theta + \theta \right] \tag{9}$$

3 Stress Results

Substituting (8) in (6) the force field is found.

$$n_{xx} = -s\left(\arctan\frac{x}{y} - \frac{xy}{x^2+y^2} \right) \tag{10}$$

$$n_{yy} = s\left(\tfrac{1}{2}\pi - \arctan\frac{x}{y} - \frac{xy}{x^2+y^2} \right) \tag{11}$$

$$n_{xy} = -s\frac{x^2}{x^2+y^2} \tag{12}$$

These are plotted in Figure 2, 3 and 4. The plate forces are independent of the distance r to the vertex.

$$n_{xx} = -s\left(\tfrac{1}{2}\pi - \theta - \cos\theta\sin\theta \right) \tag{13}$$

$$n_{yy} = s(\theta - \sin\theta\cos\theta) \tag{14}$$

$$n_{xy} = -s\cos^2\theta \tag{15}$$

The principal stresses σ_1 and σ_2 are calculated by

$$\begin{aligned} \sigma_1 &= \frac{1}{t}\left(\tfrac{1}{2}(n_{xx}+n_{yy}) + \sqrt{\tfrac{1}{4}(n_{xx}-n_{yy})^2 + n_{xy}^2} \right), \\ \sigma_2 &= \frac{1}{t}\left(\tfrac{1}{2}(n_{xx}+n_{yy}) - \sqrt{\tfrac{1}{4}(n_{xx}-n_{yy})^2 + n_{xy}^2} \right), \end{aligned} \tag{16}$$

which can be evaluated to

$$\sigma_1 = \frac{s}{t}\left(\theta - \tfrac{1}{4}\pi + \tfrac{1}{4}\sqrt{\pi^2 - 8\pi\cos\theta\sin\theta + 16\cos^2\theta}\right),$$

$$\sigma_2 = \frac{s}{t}\left(\theta - \tfrac{1}{4}\pi - \tfrac{1}{4}\sqrt{\pi^2 - 8\pi\cos\theta\sin\theta + 16\cos^2\theta}\right). \tag{17}$$

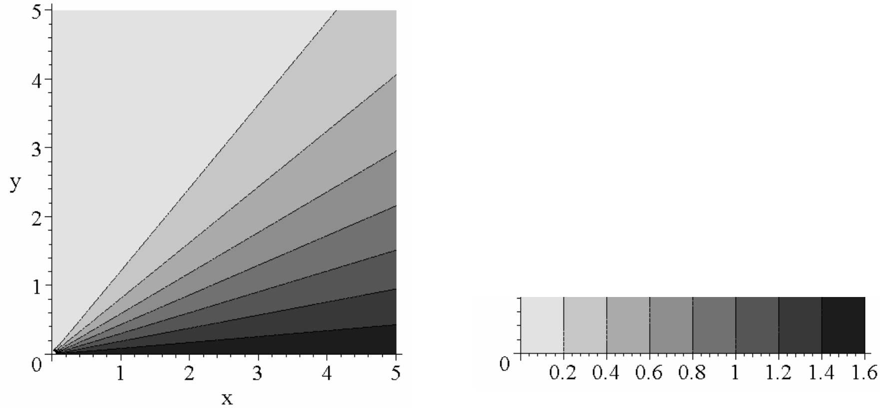

Figure 2: Normal forces $\dfrac{-n_{xx}}{s}$

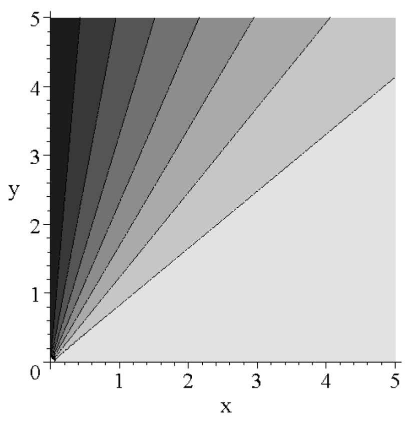

Figure 3: Normal forces $\dfrac{n_{yy}}{s}$

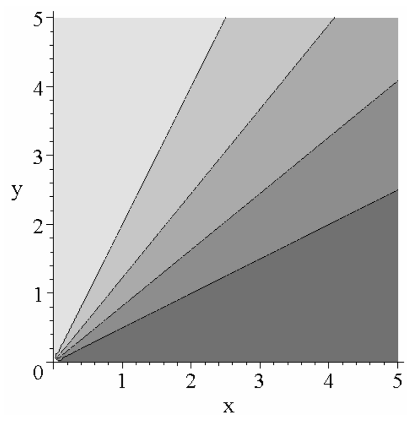

Figure 4: Shear forces $\dfrac{-n_{xy}}{s}$

The Von Misses stress is calculated by

$$\sigma_{VM} = \frac{1}{t}\sqrt{n_{xx}^2 - n_{xx}n_{yy} + n_{yy}^2 + 3n_{xy}^2}\ , \tag{18}$$

which can be evaluated to

$$\sigma_{VM} = \frac{s}{t}\frac{1}{2}\sqrt{\pi^2 - 2\pi\theta + 4\theta^2 - 6\pi\cos\theta\sin\theta + 12\cos^2\theta}\ . \tag{19}$$

All stresses are plotted in Figure 5.

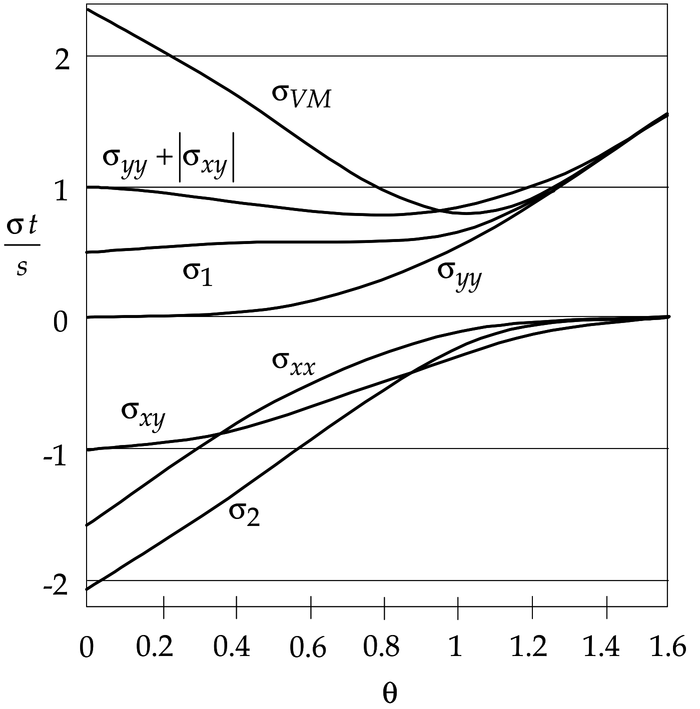

Figure 5: Stress quantities as a function of θ

4 Conclusions and Remarks

A closed form solution has been found for the stresses close to a rectangular vertex of linear elastic plate loaded by an evenly distributed shear force on one of the edges. This closed form solution was earlier derived by E. Reissner and published in a two-page technical note (Reissner 1944).

As pointed out (Reissner 1944) the solution shows that the stress tensor can be asymmetrical also in the absence of body moments. (Including body moments would nowadays be referred to as a Cosserat continuum.) After all, in the vertex the shear stresses on perpendicular faces are not the same. However, this has no practical consequences. The very concept of stress breaks down when we strongly zoom in on the material. At a very large magnification we would see individual molecules and the effects of forces between them. It would be an unrealistic idealisation to require a valid stress tensor for the single molecule that resides in the plate vertex. In other words, it is not a problem if a plate solution does not fulfil all governing equations in a finite number of points. Neither do we need to extent the concept of stress to include asymmetrical stress tensors.

In the vertex the stress is not unique. It depends on the direction in which the vertex is approached. This singularity cannot be computed easily by the finite element method because jumps are not included in the shape functions of regular elements. An extremely fine element mesh can be used to approximate the singularity. The problem of this paper is particularly suitable for bench marking finite element software because despite the singularity none of the quantities (displacements, strains, stresses) go to infinity.

Due to modern mathematical software the classical force method is very suitable to find closed form solutions of many as yet unsolved problems in elasticity theory. For example, the problem of Figure 6 has a considerable a shear lag. It can be observed in the bottom plate of a box-girder bridge or a core wall of a high-rise building. As far as the author knows, the closed form solution for that problem is still waiting to be found.

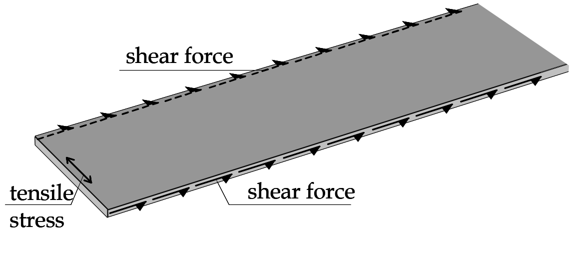

Figure 6: Shear force loading on the edges of a semi-infinite strip

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